

On hereditary properties of the class of graphs with convex quadratic stability number

Vadim V. Lozin (lozin@unn.ac.ru)

University of Nizhny Novgorod, Gagarina 23, Nizhny Novgorod, 603600 - Russia

Domingos M. Cardoso (dcardoso@mat.ua.pt)

Universidade de Aveiro, Departamento de Matemática, 3810-193 Aveiro, Portugal

Abstract. We show that the class of graphs with quadratic stability number is not hereditary. Then we prove that this class contains a unique maximal hereditary subclass, and finally, we characterize this subclass by two forbidden induced subgraphs.

Keywords: Graph theory; hereditary classes; stability number.

1. Introduction

In this report the graphs are undirected simple graphs G , for which $V(G)$ denotes the set of vertices and $E(G)$ the set of edges. An element of $E(G)$, whose ends are vertices i and j , is denoted (i, j) . It is also assumed that G is of order n (i.e., $|V(G)| = n$). Throughout the paper, K_n , P_n , and C_n will denote the complete graph, the chordless path and the chordless cycle with n vertices, respectively, $K_{n,m}$ the complete bipartite graph with two null subgraphs (subgraphs with no edges) of order n and m , A_G the adjacency matrix of G , that is, $A_G = (a_{ij})_{n \times n}$ is such that

$$a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E(G), \\ 0 & \text{otherwise,} \end{cases}$$

$P_{A_G}(\lambda)$ the characteristic polynomial of A_G , $\sigma(A_G)$ the spectrum of A_G , $\lambda_{\min}(A_G)$ the minimum eigenvalue of A_G and $\lambda_{\max}(A_G)$ the maximum eigenvalue of A_G . Given a vertex $i \in V(G)$, $N_G(i)$ will denote the neighborhood of the vertex i , that is, $N_G(i) = \{j \in V(G) : (i, j) \in E(G)\}$. Given a set of nodes U included in $V(G)$, $G-U$ will denote the graph induced by the set of nodes $V(G) \setminus U$, that is, $V(G-U) = V(G) \setminus U$ and $E(G-U) = \{(i, j) : i, j \in V(G-U), (i, j) \in E(G)\}$. For instance, if $U = \{i\}$, then the graph $G - \{i\}$ is such that $V(G - \{i\}) = V(G) \setminus \{i\}$ and $E(G - \{i\}) = E(G) \setminus \{(i, j) : j \in N_G(i)\}$. The convex quadratic programming problem

$$\max\{2e^t x - x^t(B_G + I_n)x : x \geq 0\},$$

where e^t is the all-ones row vector of \mathbf{R}^n ,

$$B_G = \begin{cases} \frac{1}{-\lambda_{\min}(A_G)} A_G, & \text{if } \lambda_{\min}(A_G) \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

and I_n is the unit matrix of order n , is denoted by (P_G) and its optimal value by $v(G)$. For a simple graph G , a subset of vertices $S \subseteq V(G)$ is stable if no two vertices in S are linked by an edge. The maximum cardinality of a stable set in G is called the stability number of G and is denoted $\alpha(G)$. As it is proved in (Luz and Cardoso, 1998), $v(G)$ is the best upper bound for $\alpha(G)$ among the optimal values of the family of quadratic programming problems

$$\phi_{a,b}^* = \max\{\phi_{a,b}(x) = ae^t x - bx^t(B_G + I_n)x : x \geq 0\},$$

with a and b such that $b \geq 0$ and $a - b \geq 1$. A graph G such that $\alpha(G) = v(G)$, as was introduced in (Cardoso and Luz, 1999), is called a graph with convex quadratic stability number and this class of graphs will be denoted by \mathcal{Q} . According to (Luz, 1995), $G \in \mathcal{Q}$ (i.e., $\alpha(G) = v(G)$) if and only if for any maximum stable set S of G ,

$$-\lambda_{\min}(A_G) \leq \min\{|N_G(i) \cap S| : i \notin S\}.$$

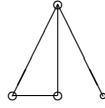


Figure 1. This graph is called Paw.

2. Introductory results

Based on the prove of the above necessary and sufficient condition for a graph to be in \mathcal{Q} , presented in (Luz, 1995), we can introduce the following result.

PROPOSITION 2.1. *If there is a maximal stable set S of G such that*

$$-\lambda_{\min}(A_G) \leq \min\{|N_G(i) \cap S| : i \notin S\},$$

then this inequality is fulfilled for every maximum stable set of G .

Proof. Let us suppose that S is a maximum stable set for which the inequality

$$-\lambda_{\min}(A_G) \leq \min\{|N_G(i) \cap S| : i \notin S\}$$

is fulfilled. Then $\alpha(G) = v(G)$ and, consequently, the characteristic vector of any maximum stable set T of G , $x(T)$, is an optimal solution for (P_G) . Therefore, by the

Karush-Kuhn-Tacker conditions, there is $y \geq 0$ such that $2(B_G + I_n)x(T) = 2e + y$. Thus, denoting by $(B_G + I_n)^i$ the i -th row of $(B_G + I_n)$, for $i \notin T$,

$$\begin{aligned} 2(B_G + I_n)^i x(T) &= 2 \frac{|N_G(i) \cap T|}{-\lambda_{\min}(A_G)} = 2 + y_i \Leftrightarrow \\ 2 \frac{|N_G(i) \cap T|}{-\lambda_{\min}(A_G)} &\geq 2 \Leftrightarrow |N_G(i) \cap T| \geq -\lambda_{\min}(A_G). \end{aligned}$$

◇

Defining

$$\rho(G) = \max\{\min\{|N_G(v) \cap S| : v \notin S\} : S \text{ is a maximum stable set of } G\},$$

we are in conditions to conclude the following corollary of proposition 2.1.

COROLLARY 2.1. *A graph G belongs to \mathcal{Q} if and only if $-\lambda_{\min}(A_G) \leq \rho(G)$.*

Proof. Suppose that $G \in \mathcal{Q}$, then, according to (Luz, 1995), for any maximum stable set T of G , $-\lambda_{\min}(A_G) \leq \min\{|N_G(i) \cap T| : i \notin T\}$. Therefore,

$$-\lambda_{\min}(A_G) \leq \rho(G).$$

Conversely, if $-\lambda_{\min}(A_G) \leq \rho(G)$, then there is a maximum stable set T of G such that $-\lambda_{\min}(A_G) \leq \min\{|N_G(i) \cap T| : i \notin T\}$. Thus, by proposition 2.1, for any maximum stable set S of G , $-\lambda_{\min}(A_G) \leq \min\{|N_G(i) \cap S| : i \notin S\}$ and hence $G \in \mathcal{Q}$. ◇

PROPOSITION 2.2. *Let G_1, G_2, \dots, G_p be the connected components of a graph G . Then*

$$v(G) = \sum_{j=1}^p v(G_j).$$

Proof. Let us denote $n_j = |V(G_j)|$ and $B_{G_j} = \frac{1}{-\lambda_{\min}(A_G)} A_{G_j}$ ($j = 1, 2, \dots, p$). Then

$$B_G + I_n = \begin{bmatrix} B_{G_1} + I_{n_1} & 0 & \dots & 0 \\ 0 & B_{G_2} + I_{n_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & B_{G_p} + I_{n_p} \end{bmatrix}.$$

Denoting by e_j the all ones vector of \mathbb{R}^{n_j} and by x_j the n_j -uple of variables, we have

$$v(G) = \max\left\{2 \sum_{j=1}^p e_j^t x_j - \sum_{j=1}^p x_j^t (B_{G_j} + I_{n_j}) x_j : x_j \geq 0\right\}$$

that is equivalent to

$$v(G) = \sum_{j=1}^p \max\{2e_j^t x_j - x_j^t (B_{G_j} + I_{n_j}) x_j : x_j \geq 0\} = \sum_{j=1}^p v(G_j).$$

◇

The following statement about the characteristic polynomials of the adjacency matrices of the graphs K_n and $K_{1,n}$ are well known, see (Cvetkovich et al, 1979).

$$P_{A_{K_n}}(\lambda) = (-1)^n(\lambda + 1)^{n-1}(\lambda - (n - 1)). \quad (1)$$

$$P_{A_{K_{1,n}}}(\lambda) = (-1)^{n-1}(\lambda^2 - n)\lambda^{n-1}. \quad (2)$$

As a direct consequence of these statements, we can conclude that $\sigma(A_{K_n}) = \{-1, n - 1\}$ and $\sigma(A_{K_{1,n}}) = \{-\sqrt{n}, 0, \sqrt{n}\}$, and therefore $\lambda_{\min}(A_{K_n}) = -1$ and $\lambda_{\min}(A_{K_{1,n}}) = -\sqrt{n}$.

It is also well known, see (Cvetkovich et al, 1979), that $\lambda_{\min}(A_G) = -1$ if and only if G is a graph with at least one edge for which every connected component is a complete graph. Furthermore, if G has at least one edge, then $-\lambda_{\max}(A_G) \leq \lambda_{\min}(A_G) \leq -1$.

PROPOSITION 2.3. *If v is a vertex of degree 1 with a neighbor u in a graph G , then*

$$P_{A_G}(\lambda) = -\lambda P_{A_{G-\{v\}}}(\lambda) - P_{A_{G-\{u,v\}}}(\lambda).$$

Proof. Let G be a graph with $V(G) = \{1, 2, \dots, n\}$ and let us suppose that v is the vertex 1 and u is the vertex 2. Then the adjacency matrix of G is

$$A_G = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & \# & \dots & \# \\ \dots & \dots & & A_{G-\{u,v\}} & \\ 0 & \# & & & \end{bmatrix},$$

where $\#$ denotes 0 or 1. Therefore,

$$\begin{aligned} P_{A_G}(\lambda) &= \det(A_G - \lambda I_n) \\ &= -\lambda \det(A_{G-\{v\}} - \lambda I_{n-1}) - \det\left(\begin{bmatrix} 1 & \# & \dots & \# \\ \dots & A_{G-\{u,v\}} - \lambda I_{n-2} & & \\ 0 & & & \end{bmatrix}\right) \\ &= -\lambda \det(A_{G-\{v\}} - \lambda I_{n-1}) - \det(A_{G-\{u,v\}} - \lambda I_{n-2}) \\ &= -\lambda P_{A_{G-\{v\}}}(\lambda) - P_{A_{G-\{u,v\}}}(\lambda). \end{aligned}$$

◇

The next proposition characterizes the class of complete q -partite graphs in terms of forbidden induced subgraphs.

PROPOSITION 2.4. *A graph G is complete q -partite if and only if G is a $\overline{K_{1,2}}$ -free graph.*

Proof. Let G be a complete q -partite graph. Then any three vertices induce in G either $\overline{K_3}$ (if all of them are in the same part) or $K_{1,2}$ (if two of them are in one part) or K_3 (if all the vertices are in different parts). Thus G does not contain $\overline{K_{1,2}}$ as an induced subgraph. Conversely, suppose that G is a $\overline{K_{1,2}}$ -free graph. Let us partition the vertex set of G into classes of vertices with the same neighborhood. Since any two adjacent vertices have different neighborhoods, every class of the partition induces in G an empty graph. Now let x and y be two vertices from different classes, and suppose that x is not adjacent to y . Since x and y have different neighborhoods, there must exist a vertex z adjacent to one of them but non adjacent to another one. But then x, y, z induce in G a $\overline{K_{1,2}}$. This contradiction proves that the classes of the partition are the parts of a complete q -partite graph. \diamond

PROPOSITION 2.5. *A graph G is (P_4, Paw) -free if and only if every connected component of G is a complete q -partite graph.*

Proof. Let G be a (P_4, Paw) -free graph and let H be a connected component of G . Suppose that vertices a, b, c induce in H a $\overline{K_{1,2}}$. Let $P(a, b)$ be a shortest (hence chordless) path linking a to b in H . Since P_4 is forbidden, this path is exactly of length 2. Let d be the unique internal vertex of the path. If d is adjacent to c , then a, b, c, d induce a Paw, that is a contradiction. If d is not adjacent to c , then a, b, c, d induce a P_4 and we get a contradiction again. Therefore H does not contain $\overline{K_{1,2}}$ as an induced subgraph and, accordingly to proposition 2.4, H is a complete q -partite graph. The converse is obvious. \diamond

3. The maximal hereditary subclass of \mathcal{Q}

A class of graphs is called hereditary if it is closed under vertex deletion. Thus, if G belongs to some hereditary class then $G - \{v\}$ also belongs to the same class for any vertex $v \in V(G)$. In order to prove that \mathcal{Q} (the class of graphs with convex quadratic stability number) is not a hereditary class, let us prove the following results.

PROPOSITION 3.1. *The chordless path on four vertices P_4 does not belong to \mathcal{Q} .*

Proof. According to proposition 2.4, we have

$$P_{AP_4}(\lambda) = -\lambda P_{AP_3}(\lambda) - P_{AP_2}(\lambda).$$

Since $P_3 = K_{1,2}$ and $P_2 = K_2$, we derive, using (1) and (2), that

$$\begin{aligned} P_{AP_4}(\lambda) &= \lambda^2(\lambda^2 - 2) - (\lambda^2 - 1) \\ &= (\lambda^2 - 1)^2 - \lambda^2 \\ &= (\lambda^2 - 1 - \lambda)(\lambda^2 - 1 + \lambda). \end{aligned}$$

Thus, $\sigma(A_{P_4}) = \{\frac{1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\}$ and, consequently,

$$\lambda_{\min}(A_{P_4}) = \frac{-1 - \sqrt{5}}{2} < -1.$$

On the other hand, it is not hard to see that $\rho(P_4) = 1$ and hence $-\lambda_{\min}(A_{P_4}) > \rho(P_4)$. Therefore, by corollary 2.1, $P_4 \notin \mathcal{Q}$. \diamond

PROPOSITION 3.2. *The graph Paw does not belong to \mathcal{Q} .*

Proof. According to proposition 2.4 and statement (1), we have

$$\begin{aligned} P_{A_{Paw}}(\lambda) &= -\lambda P_{A_{K_3}}(\lambda) - P_{A_{K_2}}(\lambda) \\ &= -\lambda(-1)^3(\lambda+1)^2(\lambda-2) - (-1)^2(\lambda+1)(\lambda-1) \\ &= (\lambda+1)(\lambda^3 - \lambda^2 - 3\lambda + 1). \end{aligned}$$

Thus, $\sigma(A_{Paw}) = \{2.170, 0.311, -1, -1.481\}$. By direct observation we conclude that $\rho(Paw) = 1$, and, consequently, $-\lambda_{\min}(A_{Paw}) > \rho(Paw)$. Therefore, $Paw \notin \mathcal{Q}$. \diamond

PROPOSITION 3.3. *The class of graphs with quadratic stability number is not a hereditary class.*

Proof. Considering the chordless path on five vertices P_5 , we have

$$\begin{aligned} P_{A_{P_5}}(\lambda) &= -\lambda P_{A_{P_4}}(\lambda) - P_{A_{P_3}}(\lambda) \\ &= -\lambda(\lambda^4 - 3\lambda^2 + 1) + \lambda(\lambda^2 - 2) \\ &= -\lambda(\lambda^2 - 1)(\lambda^2 - 3). \end{aligned}$$

Thus, $\sigma(A_{P_5}) = \{\sqrt{3}, 1, 0, -1, -\sqrt{3}\}$ and $-\lambda_{\min}(A_{P_5}) = \sqrt{3} \leq \rho(P_5) = 2$. Therefore, according to corollary 2.1, $P_5 \in \mathcal{Q}$. Consequently since, by proposition 3.1, $P_4 \notin \mathcal{Q}$, we can conclude that \mathcal{Q} is not a hereditary class. \diamond

From now on, graphs G and H such that $V(G) \cap V(H) = \emptyset$ will be called disjoint graphs, and for these graphs, $G + H$ will denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

PROPOSITION 3.4. *If disjoint graphs G and H belong to \mathcal{Q} , then $G + H \in \mathcal{Q}$.*

Proof. First note that $P_{G+H}(\lambda) = P_G(\lambda)P_H(\lambda)$, $\rho(G + H) = \max\{\rho(G), \rho(H)\}$ and $-\lambda_{\min}(A_{G+H}) = \max\{-\lambda_{\min}(A_G), -\lambda_{\min}(A_H)\}$. On the other hand, since $G \in \mathcal{Q}$ and $H \in \mathcal{Q}$, by corollary 2.1, $-\lambda_{\min}(A_G) \leq \rho(G)$ and $-\lambda_{\min}(A_H) \leq \rho(H)$. Then $\max\{-\lambda_{\min}(A_G), -\lambda_{\min}(A_H)\} \leq \max\{\rho(G), \rho(H)\}$. Hence $-\lambda_{\min}(A_{G+H}) \leq \rho(G + H)$ and thus $G + H \in \mathcal{Q}$. \diamond

PROPOSITION 3.5. *A graph $G \in \mathcal{Q}$ if and only if every connected component of G belongs to \mathcal{Q} .*

Proof. Let G_1, G_2, \dots, G_p be the connected components of G . If these connected components belong to \mathcal{Q} , then according to proposition 3.4, $G = G_1 + G_2 + \dots + G_p$ belongs to \mathcal{Q} . Conversely, let us suppose that $G = G_1 + G_2 + \dots + G_p$ belongs to \mathcal{Q} . Then, according to proposition 2.2,

$$v(G_1) + v(G_2) + \dots + v(G_p) = \alpha(G_1) + \alpha(G_2) + \dots + \alpha(G_p).$$

On the other hand, $v(G_j) \geq \alpha(G_j)$ for $j = 1, 2, \dots, p$. So $v(G_j) - \alpha(G_j) \geq 0$ for $j = 1, 2, \dots, p$ and $\sum_{j=1}^p [v(G_j) - \alpha(G_j)] = 0$, and consequently, $v(G_j) - \alpha(G_j) = 0$ for $j = 1, 2, \dots, p$. \diamond

PROPOSITION 3.6. *The unique maximal hereditary subclass of \mathcal{Q} is the class of (P_4, Paw) -free graphs.*

Proof. It is proved in (Luz, 1996) that any complete q -partite graph belongs to \mathcal{Q} . Hence, by propositions 2.5 and 3.5 2.6 and 3.5, \mathcal{Q} contains the class of (P_4, Paw) -free graphs. On the other hand, any hereditary subclass of \mathcal{Q} is contained in the class of (P_4, Paw) -free graphs, since P_4 and Paw do not belong to \mathcal{Q} . \diamond

References

- Cardoso, D.M, and Luz, C. J. The class of graphs with convex quadratic stability number. Manuscript. Group of Optimization, 1999.
- Cvetkovich, D., Doob, M., and Sachs, H. Spectra of Graphs, Theory and Applications. VEB Deutscher. Berlin, 1979.
- Luz, C.J. An upper bound on the independence number of a graph computable in polynomial time. Operations Research Letters. 18 (1995): 139-145.
- Luz, C.J. Um novo majorante para o número de estabilidade de um grafo obdito por técnicas de Programação Quadrática. PhD dissertation (in Portuguese). Universidade de Aveiro (1996).
- Luz, C.J. and Cardoso, D.M. A generalization of the Hoffman-Lovász upper bound on the independence number of a regular graph. Annals of Operations Research. 81 (1998): 307-309.

