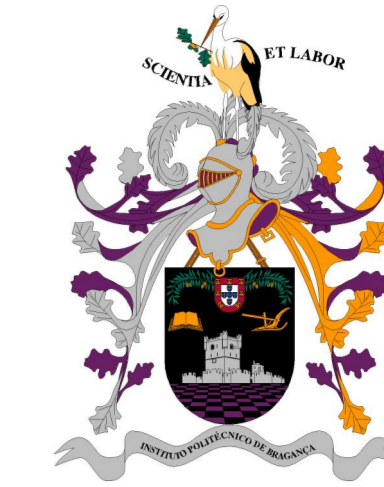


A Maple interface for computing variational symmetries in optimal control



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Abstract

A computer algebra package, for the automatic computation of variational symmetries in optimal control, was recently developed by the authors [3, 4]. Now we present a graphical user interface which permit to interact, in a point-and-click environment, with all the previous symbolical tools.

1. Introduction

The concept of variational symmetry entered into optimal control in the seventies of the twentieth century. Variational symmetries, which keep an optimal control problem invariant, are very useful in optimal control, but unfortunately their study is not easy, requiring lengthy and cumbersome calculations. Recently there has been an interest in the application of Computer Algebra Systems to the study of control systems, and collections of symbolical tools are being developed to help on the analysis and solution of complex problems. The first computer algebra package for computing the variational symmetries in the calculus of variations, and respective Noether's first integrals, was given by the authors in [2]; then extended to the more general setting of optimal control [3] and, more recently, upgraded in [4] with the introduction of new capacities, several optional parameters and improvements of efficiency. Here we provide a graphical user interface to our computer algebra package [4]. This application is named *octool* and was created with *Maplet* technology, the graphical programming language of the *Maple* system.

2. Defining Variational Symmetries and Noetherian Conservation Laws in Optimal Control

The optimal control problem consists to minimize an integral functional,

$$I[x(\cdot), u(\cdot)] = \int_a^b L(t, x(t), u(t)) dt, \quad (1)$$

subject to a control system described by a system of ODEs,

$$\dot{x}(t) = \varphi(t, x(t), u(t)), \quad (2)$$

together with appropriate boundary conditions on the values of $x(a)$ and $x(b)$. The Lagrangian $L(\cdot, \cdot, \cdot)$ is a real function, assumed to be continuously differentiable in $[a, b] \times \mathbb{R}^n \times \mathbb{R}^m$; $t \in \mathbb{R}$ the independent variable; $x: [a, b] \rightarrow \mathbb{R}^n$ the vector of state variables; $u: [a, b] \rightarrow \Omega \subseteq \mathbb{R}^m$ the vector of controls, assumed to be piecewise continuous functions; and $\varphi: [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ the velocity vector, assumed to be a continuously differentiable vector function.

Recently, the authors developed analytic computational methods [3, 4] that permits to obtain symmetries and conservation laws for a given optimal control problem (1)–(2) based on the extension of the famous Noether's theorem to optimal control [6]. Let us consider a one-parameter group of \mathbb{C}^1 transformations $h^s: [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$,

$$h^s(t, x, u, \psi) = (h_t^s(t, x, u, \psi_0, \psi), h_x^s(t, x, u, \psi_0, \psi), h_u^s(t, x, u, \psi_0, \psi), h_\psi^s(t, x, u, \psi_0, \psi)), \quad (3)$$

which reduces to the identity transformation when the parameter s vanishes: $h_t^0 = t$, $h_x^0 = x$, $h_u^0 = u$, $h_\psi^0 = \psi$. Associated with a one-parameter group of transformations (3), we introduce the infinitesimal generators

$$T(t, x, u, \psi_0, \psi) = \frac{\partial}{\partial s} h_t^s \Big|_{s=0}, \quad X(t, x, u, \psi_0, \psi) = \frac{\partial}{\partial s} h_x^s \Big|_{s=0}, \\ U(t, x, u, \psi_0, \psi) = \frac{\partial}{\partial s} h_u^s \Big|_{s=0}, \quad \Psi(t, x, u, \psi_0, \psi) = \frac{\partial}{\partial s} h_\psi^s \Big|_{s=0} \quad (4)$$

Emmy Noether was the first who established a relation between the existence of invariance transformations of the problem and the existence of conservation laws. Since the work pioneered by Noether, several definitions of invariance have been introduced for the problems of the calculus of variations and for the problems of optimal control. All these definitions are given with respect to a one-parameter group of transformations (3). Although written in a different way, one gets, in terms of the generators (4), a necessary and sufficient condition of invariance that, essentially, coincide to all those definitions. For this reason, here we define invariance directly in terms of the generators (4).

Theorem 1 (Invariance) An optimal control problem is invariant under (4) up to a gauge term $G(t, x, u, \psi_0, \psi) = \frac{d}{ds} g^s(t, x, u, \psi_0, \psi) \Big|_{s=0}$ or, equivalently, (4) is a symmetry of the problem up to G , if, and only if,

$$\frac{\partial H}{\partial t} T + \frac{\partial H}{\partial x} \cdot X + \frac{\partial H}{\partial u} \cdot U + \frac{\partial H}{\partial \psi} \cdot \Psi - \Psi^T \cdot \dot{x} - \psi^T \cdot \frac{dX}{dt} + H \frac{dT}{dt} = \frac{dG}{dt} \quad (5)$$

with H the Hamiltonian:

$$H(t, x, u, \psi_0, \psi) = \psi_0 L(t, x, u) + \psi^T \cdot \varphi(t, x, u). \quad (6)$$

A symmetry is an intrinsic property of the optimal control problem (1)–(2) (an intrinsic property of the corresponding Hamiltonian (6)), and does not depend on the Pontryagin extremals. If one restricts attention to the quadruples $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ that satisfy the Pontryagin maximum principle under presence of a nonconservative force F , one arrives to the following nonconservative version of E. Noether's theorem [1].

Theorem 2 (Nonconservative Noether's theorem) If (4) is a symmetry up to the gauge term G of problem (1)–(2) under the presence of nonconservative forces with resultant vector $F(t, x, u)$, then

$$\int (\dot{x}^T T(t, x, u, \psi_0, \psi) - X(t, x, u, \psi_0, \psi)^T \cdot F(t, x, u) dt + \psi^T \cdot X(t, x, u, \psi_0, \psi) + G(t, x, u, \psi_0, \psi) - H(t, x, u, \psi_0, \psi) T(t, x, u, \psi_0, \psi) = const \quad (7)$$

is a conservation law, i.e. (7) is valid along all the extremals.

We remark that Noetherian conservation laws (7) only depend on the generators T and X of a symmetry (4).

3. The Maple interface octool

The Maple package [4] is very general and provides a myriad of optional parameters. In this work we provide a new graphical routine, called *octool*, which permits a user to take full power of [4] without having to learn the corresponding Maple commands and optional parameters. With *octool* users can, in a point-and-click environment, interact with all the symbolical tools of our *Maple* package and deal with concrete problems of optimal control. More precisely, one has now an interface to the tree main procedures of [4]:

- i. *Symmetry*, to obtain the variational symmetries (4) and gauge term G , that satisfy the invariance condition (5);
- ii. *Noether*, to obtain the conservation laws (7); and
- iii. *PMP* (Pontryagin Maximum Principle), to try to obtain the Pontryagin extremals or, alternatively, to obtain the equations of the Hamiltonian system, stationary condition or just the Hamiltonian.

We refer the reader to [3, 4] for the definition and detailed description of these *Maple* procedures.

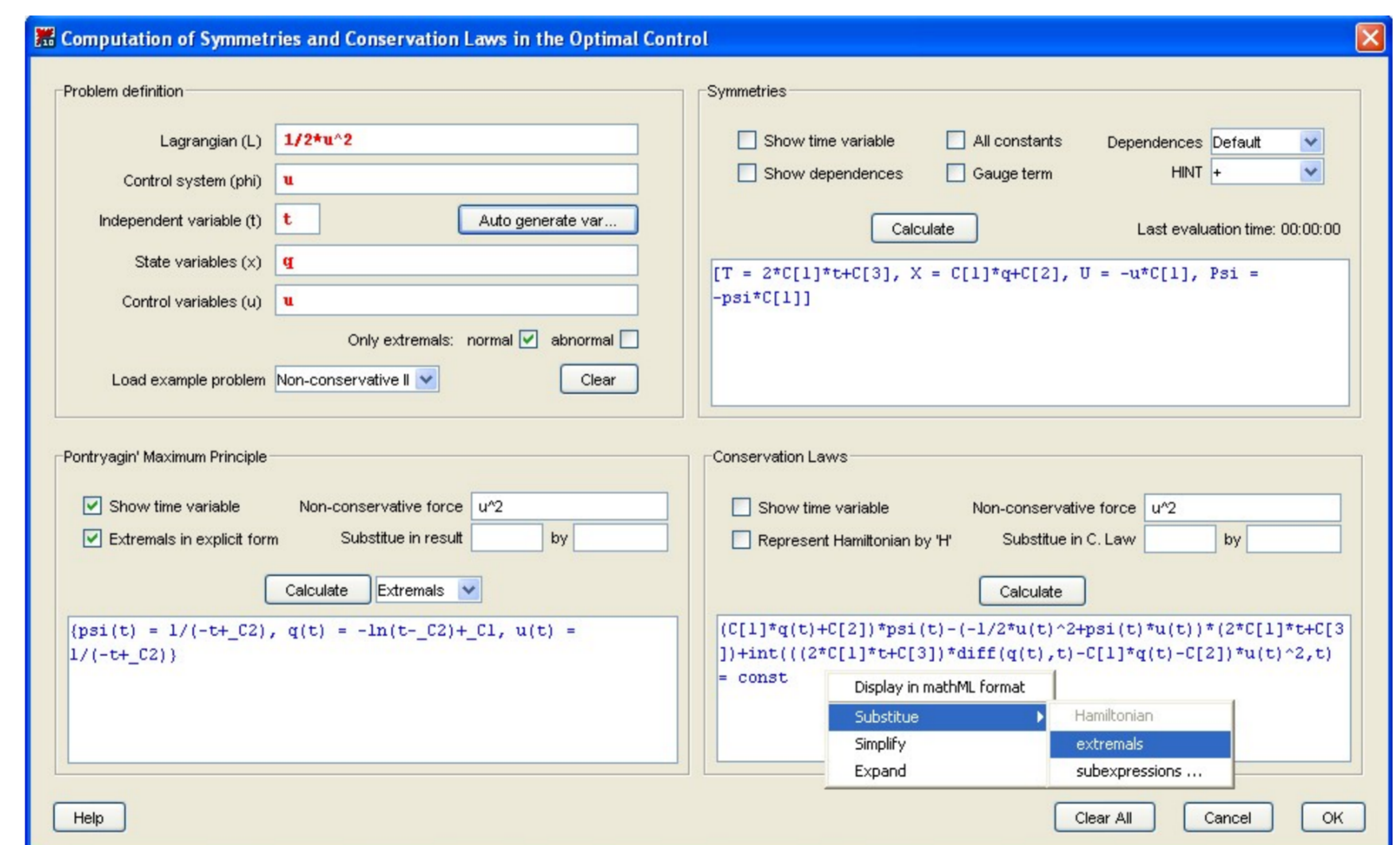


Figure 1: The octool Maple interface.

The *Maplet octool* allow us to investigate the problems in an easy and quick way, without learning all the optional parameters of the Maple procedures [3, 4]. Moreover, it permits additional algebraic manipulations. The complete Maple package, with the new procedure *octool*, can be freely obtained from

<http://www.mat.ua.pt/delfim/maple.htm>

together with many practical examples and an online help database for the Maple system.

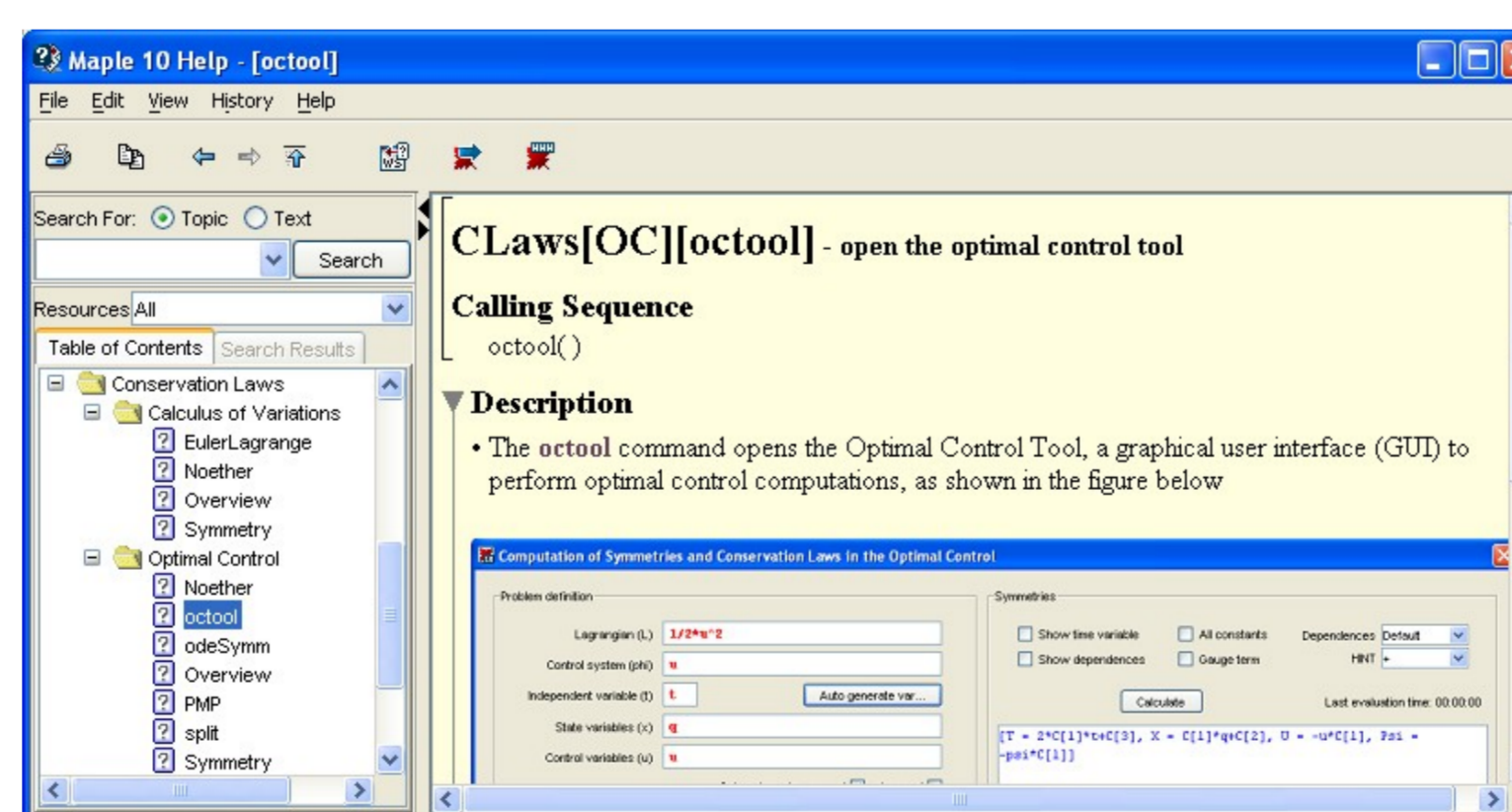
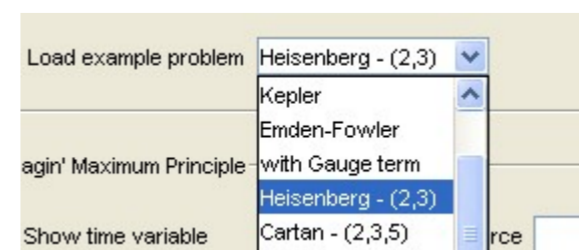
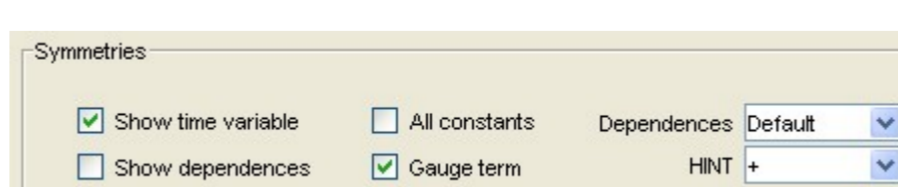


Figure 2: The octool help page for the Maple system.

Among the possibilities of the *octool*, we can point out here the following ones:

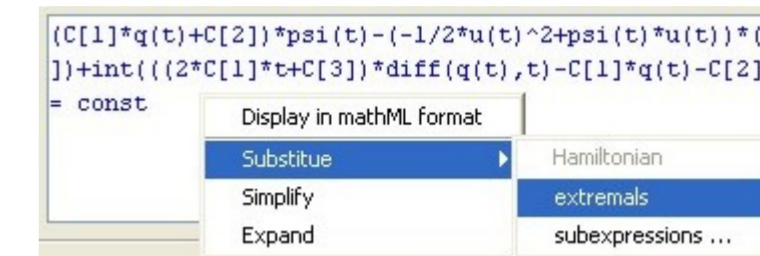
- all options can be set up using check and drop-down boxes;
- the definition of nine examples of problems of optimal control, covering different aspects, are provided, and can be automatically loaded;



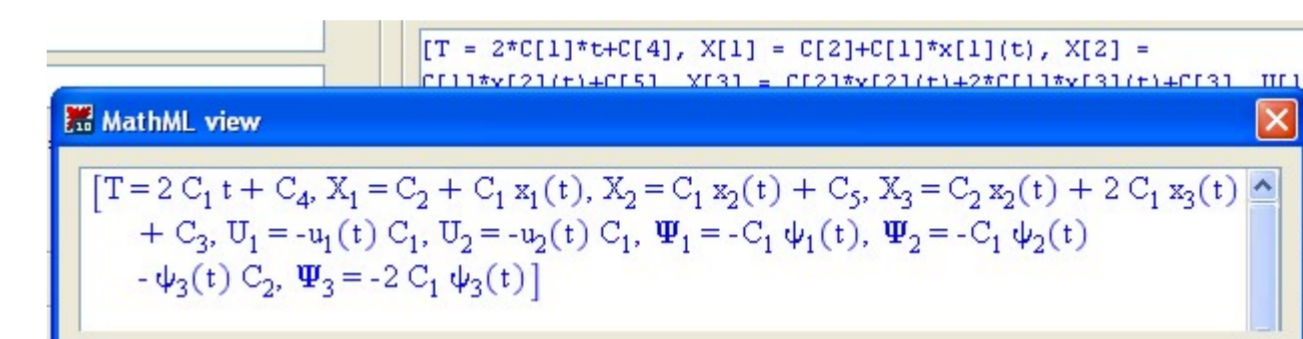
- from two check boxes one can choose to consider normal or abnormal extremals;

Only extremals: normal abnormal

- with a pop-up menu one can operate several algebraic manipulations over the results: substitutions, simplifications or expand operations;



- the display of the results can be toggled between 1-D and 2-D mathematical modes by switching between text and *MathML* formats.

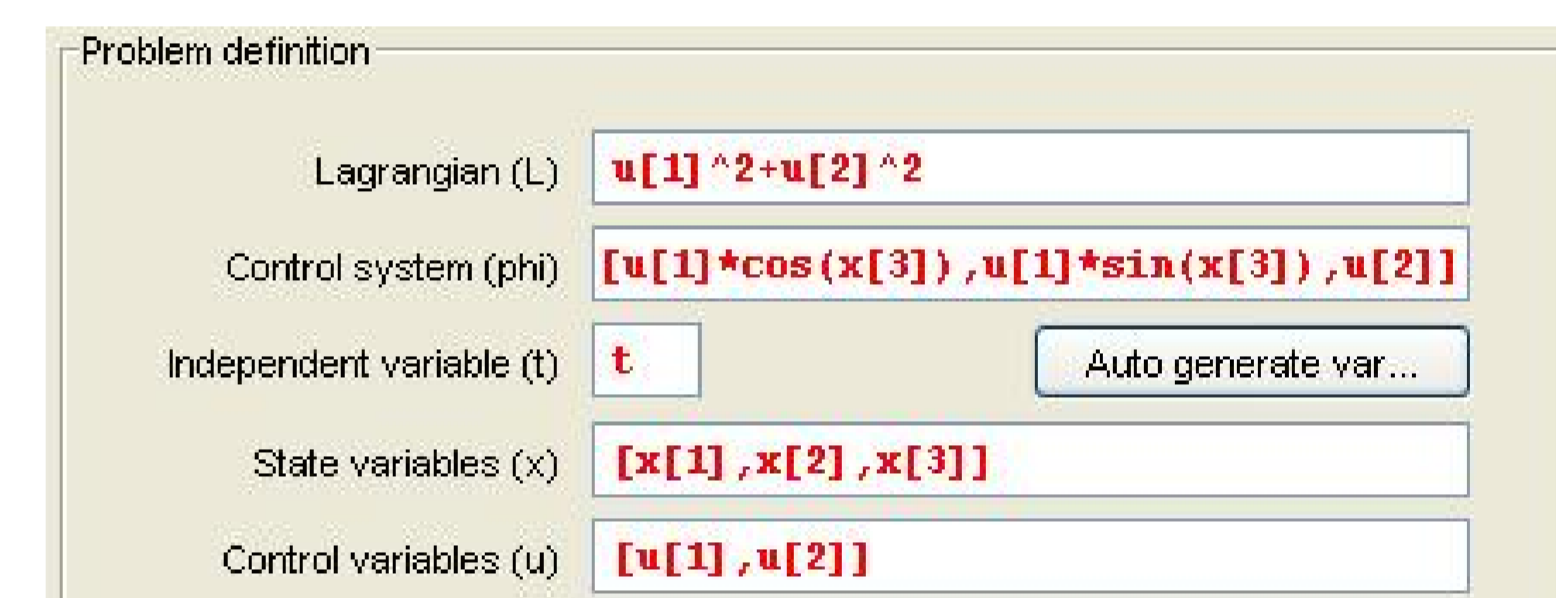


4. An illustrative example

Let us consider the following optimal control problem:

$$\int_a^b (u_1(t)^2 + u_2(t)^2) dt \rightarrow \min, \quad \begin{cases} \dot{x}_1(t) = u_1(t) \cos x_3(t), \\ \dot{x}_2(t) = u_1(t) \sin x_3(t), \\ \dot{x}_3(t) = u_2(t), \end{cases}$$

where the control system serves as model for the kinematics of a car [5, Example 18, p. 750]. We begin with the problem definition:



Our application *octool* allow us, with a simple point-and-click, to quickly determine the infinitesimal invariance generators of the optimal control problem under consideration

$$[T = C_4, X_1 = -C_1 \psi_2(t) + C_2, X_2 = C_1 \psi_1(t) + C_3, X_3 = C_1, U_1 = 0, U_2 = 0, \Psi_1 = -C_1 \psi_2(t), \Psi_2 = C_1 \psi_1(t), \Psi_3 = 0]$$

and, with another mouse click, the family of Conservation Laws associated with the generators just obtained, according with Theorem 2,

$$(-C_1 \psi_2(t) + C_2) \psi_1(t) + (C_1 \psi_1(t) + C_3) \psi_2(t) + \psi_3(t) C_1 - H C_4 = const$$

Choosing, by a pop-up menu, the substitutions

$$\{C_1=1, C_2=0, C_3=0, C_4=0\}$$

we obtain the Conservation Law

$$-\psi_2(t) \psi_1(t) + \psi_1(t) \psi_2(t) + \psi_3(t) = const$$

which corresponds to the symmetry group of planar (orientation-preserving) isometries [5, Ex. 18, p. 750].

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