Exponentiation in **V**-categories

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Abstract

For a Heyting algebra **V** which, as a category, is monoidal closed, we obtain characterizations of exponentiable objects and morphisms in the category of **V**-categories and apply them to some well-known examples. In the case $\mathbf{V} = \overline{\mathbb{R}}_+$ these characterizations of exponentiable morphisms and objects in the categories (**P**)**Met** of (pre)metric spaces and non-expansive maps show in particular that exponentiable metric spaces are exactly the almost convex metric spaces, while exponentiable complete metric spaces are the complete totally convex ones.

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0 Introduction

In 1973, Lawvere [10] observed that premetric spaces can be thought of as enriched categories over $[0,\infty]$: a premetric $d : X \times X \to [0,\infty]$ can be interpreted as the hom-functor of a category so that the inequalities

$$\begin{array}{rcl} 0 & \geq & d(x,x) \\ d(x,y)+d(y,z) & \geq & d(x,z) \end{array}$$

play the role of the unit and of the composition laws of a category.

Indeed, a **V**-category in the sense of Eilenberg and Kelly [8] is a set X endowed with a map $X \times X \to \mathbf{V}$, $(x,y) \mapsto X(x,y)$, and specifications of identity morphisms and composition law

$$\begin{array}{rcl} I & \to & X(x,x) \\ X(x,y)\otimes X(y,z) & \to & X(x,z) \end{array}$$

satisfying unity and associativity axioms, that can be expressed by commutative diagrams in the category **V**. (Here $\otimes$ is a tensor product in **V** with neutral element I.)

In our crucial example of premetric spaces, the category **V** is the complete real half-line $[0,\infty]$, with categorical structure $u \to v$ if $u \geq v$, with tensor product $+$ and unit 0. Since our category **V** is given by a lattice, commutativity of diagrams, and hence also the unity and associativity axioms, come for free.

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Another interesting example arises for $\mathbf{V}$ the two-element chain $\mathbf{2} = \{\text{false} \vdash \text{true}\}$, with the tensor product $\wedge$, that coincides with the categorical one, and neutral element true. For a map $X \times X \to \mathbf{2}$, or equivalently for a binary relation $\leq$ in X, the unit and composition laws read as

$$\begin{array}{rcl} \text{true} & \vdash & x \leq x \\ (x \leq y) \wedge (y \leq z) & \vdash & (x \leq z), \end{array}$$

so that $(X, \leq)$ is exactly a preordered set.

If $\mathbf{V} = \mathbf{Set}$, and if the tensor product is the categorical one, then a $\mathbf{V}$-category is exactly a small category.

Moreover, in the former examples the notion of $\mathbf{V}$-functor gives natural morphisms: an $\overline{\mathbb{R}}_+$-functor is a non-expansive map, a $\mathbf{2}$-functor is a monotone map, while a $\mathbf{Set}$-functor is just a functor.

The tensor product in $\mathbf{V}$ induces naturally a tensor product in the category $\mathbf{V}$-$\mathbf{Cat}$ of $\mathbf{V}$-categories and $\mathbf{V}$-functors. Lawvere proved that with $\mathbf{V}$ also $\mathbf{V}$-$\mathbf{Cat}$ is a monoidal closed category. Since in the examples $\mathbf{V} = \mathbf{2}$ and $\mathbf{V} = \mathbf{Set}$ the tensor and the categorical products coincide, this gives in particular that $\mathbf{2}$-$\mathbf{Cat}$ and $\mathbf{Cat}$(=$\mathbf{Set}$-$\mathbf{Cat}$) are cartesian closed categories. Cartesian closedness of $\overline{\mathbb{R}}_+$-$\mathbf{Cat}$ does not follow from Lawvere's result, since the tensor product $+$ does not coincide with the categorical product max. In fact this category is not cartesian closed, although $\overline{\mathbb{R}}_+$ is. Moreover, none of the categories listed is locally cartesian closed. It is possible to avoid this problem working instead in the larger category $\mathbf{V}$-$\mathbf{RGph}$ of reflexive $\mathbf{V}$-graphs, which is a quasi-topos whenever $\mathbf{V}$ is locally cartesian closed (see [5]).

The goal of this paper is to identify exponentiable morphisms and objects in $\mathbf{V}$-$\mathbf{Cat}$ when $\mathbf{V}$ is a lattice, that is, a small category with at most one morphism between each pair of objects. Then the embedding $\mathbf{V}\text{-}\mathbf{Cat} \hookrightarrow \mathbf{V}\text{-}\mathbf{RGph}$ is full, a property that plays an essential role in this work. Our main result is Theorem 3.4 which characterizes exponentiable morphisms (hence in particular exponentiable objects) in $\mathbf{V}$-$\mathbf{Cat}$ whenever the lattice $\mathbf{V}$ is complete and locally cartesian closed, which means exactly that $\mathbf{V}$ is a complete Heyting algebra. This result gives in particular the characterization of exponentiable monotone maps in $\mathbf{POrd}$ obtained by Tholen [12] and new characterizations of exponentiable morphisms and objects in the category $\mathbf{Met}$ of metric spaces and non-expansive maps.

Finally we observe that some of the proofs presented in this paper follow the guidelines of the proof of the characterization of exponentiable continuous maps in $\mathbf{Top}$ obtained in [4], which raises the problem of whether these techniques can be applied to obtain such a characterization in the more general context of reflexive and transitive lax algebras described in [3] or [6].

In Section 1 we present the basic results on $\mathbf{V}$-categories we need throughout. Section 2 gives an account on exponentiability, including exponentiation in the category of reflexive $\mathbf{V}$-graphs. Section 3 contains our main results, namely the characterizations of exponentiable morphisms and objects in $\mathbf{V}$-$\mathbf{Cat}$. The last section deals with examples, giving a special attention to exponentiability in the category $\mathbf{Met}$ of metric spaces and non-expansive maps.

1 The category of V-categories

1.1 Throughout $\mathbf{V}$ is a complete lattice equipped with a symmetric tensor product $\otimes$, with unit k, and with right adjoint hom; that is, for each $u, v, w \in \mathbf{V}$,

$$u \otimes v \le w \ \Leftrightarrow\ v \le \hom(u, w).$$

As a category, $\mathbf{V}$ is said to be a *closed symmetric monoidal category.*

A $\mathbf{V}$-*enriched category* (or simply $\mathbf{V}$-*category*) is a pair (X, a) with X a set and $a : X \times X \to \mathbf{V}$ a map such that:

(R) for each $x \in X$, $k \le a(x, x)$;

(T) for each $x, x', x'' \in X$, $a(x, x') \otimes a(x', x'') \le a(x, x'')$.

Given $\mathbf{V}$-categories (X, a) and (Y, b), a $\mathbf{V}$-*functor* $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that, for each $x, x' \in X$, $a(x, x') \le b(f(x), f(x'))$. They form the category $\mathbf{V}$-$\mathbf{Cat}$ of $\mathbf{V}$-categories and $\mathbf{V}$-functors.

1.2 We list now two guiding examples of such categories, obtained when $\mathbf{V}$ is the two-element chain $\mathbf{2} = \{\text{false} \vdash \text{true}\}$, with the monoidal structure given by $\wedge$ and "true", and when $\mathbf{V}$ is the (complete) real half-line $[0, \infty]$, with the categorical structure induced by the relation $\ge$ (i.e., $a \to b$ means $a \ge b$) and with tensor product $+$, which we will denote by $\overline{\mathbb{R}}_+$.

For $\mathbf{V} = \mathbf{2}$, with the usual notation $x \le x' :\Leftrightarrow a(x, x') = \text{true}$, axioms (R) and (T) read as:

$$\begin{array}{lrcl} \forall x \in X & x & \le & x; \\ \forall x, x', x'' \in X & x \le x' \ \&\ x' \le x'' & \Rightarrow & x \le x'' \end{array}$$

that is, $(X, \le)$ is a preordered set. A $\mathbf{2}$-functor is then a map $f : (X, \le) \to (Y, \le)$ between preordered sets such that

$$\forall x, x' \in X \qquad x \le x' \ \Rightarrow\ f(x) \le f(x');$$

that is, f is a monotone map. Hence $\mathbf{2}$-$\mathbf{Cat}$ is exactly the category $\mathbf{POrd}$ of preordered sets and monotone maps.

An $\overline{\mathbb{R}}_+$-category is a set X endowed with a map $a : X \times X \to [0, \infty]$ such that

$$\begin{array}{lrcl} \forall x \in X & 0 & \ge & a(x, x); \\ \forall x, x', x'' \in X & a(x, x') + a(x', x'') & \ge & a(x, x''); \end{array}$$

that is, $a : X \times X \to [0, \infty]$ is a premetric in X. An $\overline{\mathbb{R}}_+$-functor is a map $f : (X, a) \to (Y, b)$ between premetric spaces satisfying the following inequality:

$$\forall x, x' \in X \qquad a(x, x') \ge b(f(x), f(x')),$$

which means that f is a non-expansive map. Therefore the category $\overline{\mathbb{R}}_+$-**Cat** coincides with the category **PMet** of premetric spaces and non-expansive maps. (For more details, see [10, 6].)

1.3 The **V**-functor hom : $\mathbf{V} \times \mathbf{V} \to \mathbf{V}$ provides **V** with a structure of **V**-category. Indeed, for each $u, v, w \in \mathbf{V}$:

$$\begin{array}{rcl} k & \leq & \hom(u,u) \\ \hom(u,v) \otimes \hom(v,w) & \leq & \hom(u,w). \end{array}$$

The first inequality follows from $k \otimes u = u \leq u$, while the second is a consequence of

$$u \otimes \hom(u,v) \otimes \hom(v,w) \leq v \otimes \hom(v,w) \leq w.$$

1.4 The tensor product $\otimes$ of **V** induces a tensor product in **V**-**Cat**, which we will denote also by $\otimes$: for each pair $(X,a), (Y,b)$ of **V**-categories, $(X,a) \otimes (Y,b) := (X \times Y, a \otimes b)$, where $(a \otimes b)((x,y),(x',y')) = a(x,x') \otimes b(y,y')$; it is clear that, for each pair of **V**-functors $f : (X,a) \to (Y,b)$ and $g : (X',a') \to (Y',b')$, the map $f \times g : X \times X' \to Y \times Y'$ becomes a **V**-functor between the corresponding **V**-categories.

This tensor product has a unit element, $K = (\{*\}, k)$, a singleton set with $k(*,*) = k$. Moreover, it was shown in [10]:

1.5 Theorem. *The category* **V**-**Cat** *equipped with the tensor product* $\otimes$ *is a closed category.*

The description of its right adjoint hom, constructed in [10], becomes very simple in this context: for each pair (X,a) and (Y,b) of **V**-categories, $\hom((X,a),(Y,b))$ has as underlying set $X^Y = \{f : (X,a) \to (Y,b)\,;\, f \text{ is a } \mathbf{V}\text{-functor}\}$, with structure d defined by:

$$d(f,g) = \bigwedge_{x \in X} b(f(x), g(x)).$$

1.6 In case $\otimes$ coincides with the categorical product, as it is the case when $\mathbf{V} = \mathbf{2}$, the category of **V**-categories is automatically cartesian closed by the previous theorem. Whenever $\otimes$ is not the categorical product, this result is no longer valid. For instance, **PMet**, studied in detail in the last section (see 4.2), is not cartesian closed.

Even in the case $\otimes$ is the categorical product, cartesian closedness is not inherited by the slice categories: for instance, **POrd** is not locally cartesian closed (see [12] and Theorem 4.1).

2 The category of reflexive V-graphs

2.1 If we drop the axiom (T) in the definition of **V**-category, we end up with the *category* **V**-**RGph** *of reflexive* **V**-*graphs and* **V**-*functors*. That is, objects of **V**-**RGph** are pairs (X,a), where X is a set and $a : X \times X \to \mathbf{V}$ is a map such that $k \leq a(x,x)$ for every $x \in X$; morphisms $f : (X,a) \to (Y,b)$ in **V**-**RGph** are maps $f : X \to Y$ such that $a(x,x') \leq b(f(x), f(x'))$, for all $x, x' \in X$.

This category contains **V**-**Cat** as a full subcategory, and, moreover, as it was shown in [3]:

2.2 Theorem. *In the commutative diagram*

$$\begin{array}{ccc} \mathbf{V}\text{-}\mathbf{Cat} & \hookrightarrow & \mathbf{V}\text{-}\mathbf{RGph} \\ & \searrow \quad \swarrow & \\ & \mathbf{Set} & \end{array}$$

the (full) embedding is reflective, with identity maps as reflections, and the forgetful functors are topological. □

In contrast with the situation in **V**-**Cat**, exponentiation in **V**-**RGph** is easily described, based on exponentiation in **Set** and **V**, as developed in [5].

2.3 For the forthcoming study of exponentiation in **V**-**Cat**, the description of exponentials in **V**-**RGph** is crucial. Our first observation in this direction is that the use of partial products as introduced in [7] turns out to simplify this study. Next we summarize the results needed for that.

2.4 Theorem. ([11, 7].) *For a morphism $f : X \to Y$ in a category* **C***, the following conditions are equivalent:*

(i) *f is exponentiable, i.e. $f \times - : \mathbf{C}/Y \to \mathbf{C}/Y$ has a right adjoint;*

(ii) *the 'change-of-base' functor $f \times_Y - : \mathbf{C}/Y \to \mathbf{C}/X$ has a right adjoint;*

(iii) *the 'pullback' functor $X \times_Y - : \mathbf{C}/Y \to \mathbf{C}$ has a right adjoint;*

(iv) **C** *has partial products over f.* □

We recall that **C** *has partial products over* $f : X \to Y$ if, for each object Z in **C**, there is a diagram

$$\begin{array}{ccccc} Z & \xleftarrow{\text{ev}} & X \times_Y P & \xrightarrow{\pi_1} & X \\ & & {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle f} \\ & & P & \xrightarrow{p} & Y \end{array}$$

such that every diagram

$$\begin{array}{ccccc} Z & \xleftarrow{\text{ev}'} & X \times_Y Q & \xrightarrow{\pi_1'} & X \\ & & {\scriptstyle \pi_2'}\downarrow & & \downarrow{\scriptstyle f} \\ & & Q & \xrightarrow{q} & Y \end{array}$$

factors as $p \cdot t = q$ and $\text{ev} \cdot (1_X \times t) = \text{ev}'$ by a unique morphism $t : Q \to P$.

In [5] it is shown that, when **V** is a locally cartesian category – which, in our situation just means **V** is a Heyting algebra – **V**-**RGph** has partial products over every **V**-functor $f : (X, a) \to (Y, b)$. In fact:

2.5 Theorem. ([5].) *If* **V** *is a Heyting algebra, then* **V**-**RGph** *is a quasitopos.* □

Here we will sketch only the construction of partial products, since they will play a crucial role in the subsequent study.

2.6 For each **V**-functor $f : (X, a) \to (Y, b)$ between reflexive **V**-graphs and for each reflexive **V**-graph (Z, c), the partial product (P, d) of (Z, c) over f is defined as follows:

$$P = \{(s, y)\,;\, y \in Y \text{ and } s : (f^{-1}(y), a) \to (Z, c) \text{ is a } \mathbf{V}\text{-functor}\}$$

(here $a : f^{-1}(y) \times f^{-1}(y) \to \mathbf{V}$ is just the restriction of $a : X \times X \to \mathbf{V}$); to obtain the structure d on P, for each $(s, y), (s', y') \in P$, $x \in f^{-1}(y)$ and $x' \in f^{-1}(y')$, we first form in **V** the partial product $v((s, x), (s', x'))$ of $c(s(x), s'(x'))$ over $a(x, x') \le b(y, y')$; then $d((s, y), (s', y'))$ is the multiple pullback of the $(v((s, x), (s', x')))_{x\in f^{-1}(y),\, x'\in f^{-1}(y')}$ in the lattice $\downarrow b(y, y')$; that is

$$d((s,y),(s',y')) \;=\; \begin{cases} \displaystyle\bigwedge_{x\in f^{-1}(y),\, x'\in f^{-1}(y')} v((s,x),(s',x')), & \text{if } f^{-1}(y) \neq \emptyset \neq f^{-1}(y'), \\ b(y,y'), & \text{otherwise.} \end{cases}$$

Since

$$v((s,x),(s',x')) = \bigwedge_{x\in f^{-1}(y),\, x'\in f^{-1}(y')} \bigvee\{v \in \mathbf{V}\,;\, v \le b(y,y') \text{ and } a(x,x') \wedge v \le c(s(x), s'(x'))\},$$

using the distributivity of binary meets over arbitrary joins, we obtain the following description of $d((s, y), (s', y'))$, which is easier to handle is this context:

$$d((s,y),(s',y')) = \max\left\{ v \in \mathbf{V}\,;\, \begin{array}{l} v \le b(y,y')\ \& \\ \forall x \in f^{-1}(y)\, \forall x' \in f^{-1}(y')\ a(x,x') \wedge v \le c(s(x), s'(x')) \end{array} \right\}.$$

3 Exponentiation in V-Cat

3.1 In order to characterize exponentiable morphisms, and consequently exponentiable objects, in **V-Cat**, we first state some auxiliary results.

3.2 Lemma. *The following conditions are equivalent:*

(i) *k is the top element of* **V***;*

(ii) *for every $u, v, w \in \mathbf{V}$,*

$$(u \wedge v) \otimes w \le (u \otimes w) \wedge v. \qquad \square$$

3.3 Proposition. *If* **V** *is a Heyting algebra, k is the top element of* **V***, $f : (X, a) \to (Y, b)$ is an exponentiable morphism in* **V-Cat** *and (Z, c) is a* **V***-category, then the partial products of (Z, c) over f in* **V-Cat** *and in* **V-RGph** *coincide.*

Proof. Let

$$\begin{array}{ccccc} (Z,c) & \xleftarrow{\ \mathrm{ev}\ } & (X \times_Y P, \tilde{d}) & \xrightarrow{\ \pi_1\ } & (X,a) \\ & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle f} \\ & & (P,d) & \xrightarrow{\ p\ } & (Y,b) \end{array} \quad \text{and} \quad \begin{array}{ccccc} (Z,c) & \xleftarrow{\ \mathrm{ev}'\ } & (X \times_Y P', \tilde{d}') & \xrightarrow{\ \pi_1'\ } & (X,a) \\ & & \downarrow{\scriptstyle \pi_2'} & & \downarrow{\scriptstyle f} \\ & & (P',d') & \xrightarrow{\ p'\ } & (Y,b) \end{array}$$

be the partial products of (Z, c) over f in **V-RGph** and **V-Cat**, respectively. Since **V-Cat** is closed under pullbacks and the latter diagram lies in particular in **V-RGph**, there exists a unique **V**-functor $t : (P', d') \to (P, d)$ such that $p \cdot t = p'$ and $\mathrm{ev} \cdot (1 \times t) = \mathrm{ev}'$. It is easy to check that $t : P' \to P$ is a bijection, using the universal properties of both diagrams. We therefore assume, for simplicity, that t is an identity map. To show it is an isomorphism, that is, $d = d'$, let $(s_0, y_0), (s_1, y_1) \in P$. The pair $(\{0, 1\}, e)$ with $e(0, 1) = d((s_0, y_0), (s_1, y_1))$ and $e(0, 0) = e(1, 1) = k$ is a **V**-category. Hence, the diagram in **V-Cat**

$$\begin{array}{ccccc} (Z,c) & \xleftarrow{\mathrm{ev}\cdot\iota} & (X \times_Y \{0,1\}, \tilde{e}) & \longrightarrow & (X,a) \\ & & \downarrow & & \downarrow f \\ & & (\{0,1\}, e) & \xrightarrow{p\cdot\iota} & (Y,b), \end{array}$$

where $\iota : (\{0, 1\}, e) \to (P, d)$ is the inclusion map, with $\iota(0) = (s_0, y_0)$ and $\iota(1) = (s_1, y_1)$, induces a **V**-functor $(\{0, 1\}, e) \to (P', d')$ by the universal property of (P', d'). Hence $d((s_0, y_0), (s_1, y_1)) = e(0, 1) \leq d'((s_0, y_0), (s_1, y_1))$ and the conclusion follows. □

3.4 Theorem. *If* **V** *is a Heyting algebra and k is the top element of* **V***, then a* **V***-functor $f : (X, a) \to (Y, b)$ is exponentiable in* **V-Cat** *if and only if, for each $x_0, x_2 \in X$, $y_1 \in Y$ and for each $v_0, v_1 \in$* **V** *such that $v_0 \leq b(f(x_0), y_1)$ and $v_1 \leq b(y_1, f(x_2))$,*

$$\bigvee_{x \in f^{-1}(y_1)} (a(x_0, x) \wedge v_0) \otimes (a(x, x_2) \wedge v_1) \geq a(x_0, x_2) \wedge (v_0 \otimes v_1). \qquad (*)$$

Proof. We first show that condition $(*)$ above guarantees the exponentiability of f. More precisely, we show that, for any **V**-category (Z, c), its partial product (P, d) over f, formed in **V-RGph**, is a **V**-category; that is, it satisfies axiom (T):

$$\forall (s_0, y_0), (s_1, y_1), (s_2, y_2) \in P \;\; d((s_0, y_0), (s_1, y_1)) \otimes d((s_1, y_1), (s_2, y_2)) \leq d((s_0, y_0), (s_2, y_2)).$$

Let $u_0 := d((s_0, y_0), (s_1, y_1))$, $u_1 := d((s_1, y_1), (s_2, y_2))$ and $u := d((s_0, y_0), (s_2, y_2))$. For every $x_i \in f^{-1}(y_i)$, $i = 0, 1, 2$, we have

$$u_0 \leq b(y_0, y_1) \text{ and } a(x_0, x_1) \wedge u_0 \leq c(s_0(y_0), s_1(y_1)),$$

$$u_1 \leq b(y_1, y_2) \text{ and } a(x_1, x_2) \wedge u_1 \leq c(s_1(y_1), s_2(y_2)).$$

Hence, for every $x_0 \in f^{-1}(y_0)$ and $x_2 \in f^{-1}(y_2)$,

$$\begin{aligned} u_0 \otimes u_1 \leq b(y_0, y_1) \otimes b(y_1, y_2) &\leq b(y_0, y_2), \text{ and} \\ a(x_0, x_2) \wedge (u_0 \otimes u_1) &\leq \bigvee_{x \in f^{-1}(y_1)} (a(x_0, x) \wedge u_0) \otimes (a(x, x_2) \wedge u_1) \\ &\leq \bigvee_{x \in f^{-1}(y_1)} c(s_0(x_0), s_1(x)) \otimes c(s_1(x), s_2(x_2)) \\ &\leq c(s_0(x_0), s_2(x_2)). \end{aligned}$$

Therefore, $u_0 \otimes u_1 \leq u$ as claimed.

To show the necessity of the condition, we consider $x_0, x_2 \in X$, $y_0 = f(x_0)$, $y_2 = f(x_2)$, $y_1 \in Y$, $v_0, v_1 \in \mathbf{V}$ as in $(*)$, and we define a triple of maps

$$\begin{array}{rcl} s_0 : f^{-1}(y_0) & \to & \mathbf{V} \\ x & \mapsto & a(x_0, x), \\ s_1 : f^{-1}(y_1) & \to & \mathbf{V} \\ x & \mapsto & a(x_0, x) \wedge v_0, \\ s_2 : f^{-1}(y_2) & \to & \mathbf{V} \\ x & \mapsto & \displaystyle\bigvee_{x_1 \in f^{-1}(y_1)} (a(x_0, x_1) \wedge v_0) \otimes (a(x_1, x) \wedge v_1). \end{array}$$

The map s_0 is a $\mathbf{V}$-functor, since, for each $x, x' \in f^{-1}(y_0)$,

$$a(x_0, x) \otimes a(x, x') \leq a(x_0, x') \;\Rightarrow\; a(x, x') \leq \hom(a(x_0, x), a(x_0, x')) = \hom(s_0(x), s_0(x')).$$

In order to check that s_1 is a $\mathbf{V}$-functor, let $x, x' \in f^{-1}(y_1)$. Then

$$s_1(x) \otimes a(x, x') = (a(x_0, x) \wedge v_0) \otimes a(x, x') \leq (a(x_0, x) \otimes a(x, x')) \wedge v_0 \leq a(x_0, x') \wedge v_0 = s_1(x')$$

$$\Rightarrow\; a(x, x') \leq \hom(s_1(x), s_1(x')).$$

Finally we have to check that s_2 is also a $\mathbf{V}$-functor: for $x, x' \in f^{-1}(y_2)$,

$$\begin{array}{rcl} s_2(x) \otimes a(x, x') & = & \displaystyle\bigvee_{x_1 \in f^{-1}(y_1)} (a(x_0, x_1) \wedge v_0) \otimes (a(x_1, x) \wedge v_1) \otimes a(x, x') \\ & \leq & \displaystyle\bigvee_{x_1 \in f^{-1}(y_1)} (a(x_0, x_1) \wedge v_0) \otimes ((a(x_1, x) \otimes a(x, x')) \wedge v_1) \\ & \leq & \displaystyle\bigvee_{x_1 \in f^{-1}(y_1)} (a(x_0, x_1) \wedge v_0) \otimes (a(x_1, x') \wedge v_1) \\ & = & s_2(x'); \end{array}$$

hence, $a(x, x') \leq \hom(s_2(x), s_2(x'))$.

Finally, it is easy to check that

$$v_0 \leq d((s_0, y_0), (s_1, y_1)) \text{ and } v_1 \leq d((s_1, y_1), (s_2, y_2)),$$

since: $v_0 \leq b(y_0, y_1)$ by hypothesis, and, for every $x_0' \in f^{-1}(y_0)$, $x_1' \in f^{-1}(y_1)$,

$$a(x_0', x_1') \wedge v_0 \leq \hom(s_0(x_0'), s_1(x_1'))$$

follows from

$$\begin{array}{rcl} s_0(x_0') \otimes (a(x_0', x_1') \wedge v_0) & = & a(x_0, x_0') \otimes (a(x_0', x_1') \wedge v_0) \\ & \leq & (a(x_0, x_0') \otimes a(x_0', x_1')) \wedge v_0 \\ & \leq & a(x_0, x_1') \wedge v_0 = s_1(x_1'); \end{array}$$

analogously, $v_1 \leq b(y_1, y_2)$ by hypothesis, and, for each $x'_1 \in f^{-1}(y_1)$, $x'_2 \in f^{-1}(y_2)$,

$$a(x'_1, x'_2) \wedge v_1 \leq \hom(s_1(x'_1), s_2(x'_2))$$

follows easily from the inequalities

$$\begin{array}{rcl} s_1(x'_1) \otimes (a(x'_1, x'_2) \wedge v_1) & = & (a(x_0, x'_1) \wedge v_0) \otimes (a(x'_1, x'_2) \wedge v_1) \\ & \leq & \displaystyle\bigvee_{x_1 \in f^{-1}(y_1)} (a(x_0, x_1) \wedge v_0) \otimes (a(x_1, x'_2) \wedge v_1) = s_2(x'_2). \end{array}$$

Therefore, since we are assuming that (P, d) is a $\mathbf{V}$-category, we may conclude that

$$d((s_0, y_0), (s_2, y_2)) \geq v_0 \otimes v_1,$$

which means in particular that

$$a(x_0, x_2) \wedge (v_0 \otimes v_1) \leq \hom(s_0(x_0), s_2(x_2)) = \hom(k, s_2(x_2)) = s_2(x_2);$$

that is,

$$a(x_0, x_2) \wedge (v_0 \otimes v_1) \leq \bigvee_{x_1 \in f^{-1}(y_1)} (a(x_0, x_1) \wedge v_0) \otimes (a(x_1, x_2) \wedge v_1),$$

which completes the proof. □

3.5 Corollary. *If $\mathbf{V}$ is a Heyting algebra with k its top element, then a $\mathbf{V}$-category (X, a) is exponentiable in $\mathbf{V}$-$\mathbf{Cat}$ if and only if*

$$\forall x_0, x_2 \in X \; \forall v_0, v_1 \in \mathbf{V} \; \bigvee_{x \in X} (a(x_0, x) \wedge v_0) \otimes (a(x, x_2) \wedge v_1) \geq a(x_0, x_2) \wedge (v_0 \otimes v_1). \quad \square$$

3.6 Corollary. *If the tensor product $\otimes$ and the categorical product coincide in $\mathbf{V}$, then a $\mathbf{V}$-functor $f : (X, a) \to (Y, b)$ is exponentiable in $\mathbf{V}$-$\mathbf{Cat}$ if and only if*

$$\forall x_0, x_2 \in X \;\; \forall y_1 \in Y \quad \bigvee_{x \in f^{-1}(y_1)} (a(x_0, x) \wedge a(x, x_2)) \geq a(x_0, x_2) \wedge b(f(x_0), y_1) \wedge b(y_1, f(x_2)).$$

Proof. If $\otimes = \wedge$, one has

$$\bigvee_{x \in f^{-1}(y_1)} (a(x_0, x) \wedge v_0) \otimes (a(x, x_2) \wedge v_1) = v_0 \wedge v_1 \wedge \bigvee_{x \in f^{-1}(y_1)} (a(x_0, x) \wedge a(x, x_2)),$$

and

$$\begin{array}{rl} & v_0 \wedge v_1 \wedge \displaystyle\bigvee_{x \in f^{-1}(y_1)} (a(x_0, x) \wedge a(x, x_2)) \geq a(x_0, x_2) \wedge v_0 \wedge v_1 \\ \Leftrightarrow & \displaystyle\bigvee_{x \in f^{-1}(y_1)} (a(x_0, x) \wedge a(x, x_2)) \geq a(x_0, x_2) \wedge v_0 \wedge v_1. \end{array}$$

Finally it is clear that it is enough to check the latter inequality for $v_0 = b(f(x_0), y_1)$ and $v_1 = b(y_1, f(x_2))$. $\square$

3.7 Remark. The description of quotients obtained in [9] gives an alternative way of showing that $(*)$ is sufficient for exponentiability. First, one can prove that a pullback of a quotient map $g : Z \to Y$ along f is again a quotient map. Roughly speaking, (*) guarantees that, for x_0, x_2 in X, each zigzag in Z mapped to $f(x_0)$, $f(x_2)$ can be lifted in an appropriate way to a zigzag in $X \times_Y Z$ mapped to x_0, x_2. Finally, it is easy to check that property $(*)$ is pullback-stable, hence the change of base functor $f \times_Y -$ preserves quotients, and therefore, by Freyd's Adjoint Functor Theorem, it has a right adjoint.

4 Examples

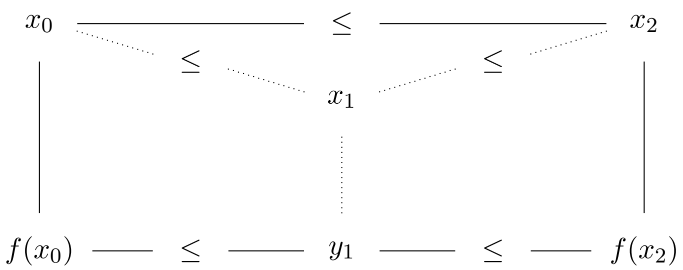
4.1 Exponentiation in POrd. We consider first exponentiation in **V-Cat** for $\mathbf{V} = \mathbf{2}$, that is, in the category **POrd** of preordered sets and monotone maps. In this category the condition of Corollary 3.6 trivializes except for $b(f(x_0), y_1) = b(y_1, f(x_2)) = a(x_0, x_2) = \text{true}$. Then it means that:

$$\bigvee_{x \in f^{-1}(y_1)} a(x_0, x) \wedge a(x, x_2) = \text{true};$$

that is, there exists $x \in X$ such that $f(x) = y_1$ and $x_0 \leq x \leq x_2$. Therefore we have:

Theorem. ([12].) *A monotone map* $f : (X, \leq) \to (Y, \leq)$ *between preordered sets is exponentiable in* **POrd** *if and only if*

$$\forall x_0, x_2 \in X\ \forall y_1 \in Y\ x_0 \leq x_2\ \&\ f(x_0) \leq y_1 \leq f(x_2) \ \Rightarrow\ \exists x_1 \in X\ :\ f(x_1) = y_1\ \&\ x_0 \leq x_1 \leq x_2 :$$



$\square$

We observe that, if $f : (X, \leq) \to (Y, \leq)$ is an embedding, this condition can be restated as:

Corollary. *An embedding* $f : (X, \leq) \to (Y, \leq)$ *is exponentiable in* **POrd** *if and only if*

$$\downarrow f(X) \cap \uparrow f(X) = f(X). \qquad \square$$

Using the closure defined by $\uparrow$, this just means that $f(X)$ is the intersection of an open and a closed subset of Y, which resembles the characterization of exponentiable embeddings in **Top** as the locally closed embeddings (see [11], Corollary 2.7).

4.2 Exponentiation in PMet. Now we consider $\mathbf{V} = \overline{\mathbb{R}}_+ = ([0, \infty], \geq)$ the (complete) real half-line. As we have already observed, the category **V-Cat** is the category **PMet** of premetric spaces and non-expansive maps. Here condition $(*)$ of Theorem 3 may be also simplified, as we explain next. In the sequel we use the natural order in the real numbers, which is the opposite of the categorical one, so that $(*)$ reads as: for each $x_0, x_2 \in X$, $y_1 \in Y$, and for each $v_0, v_1 \in \overline{\mathbb{R}}_+$ such that $v_0 \geq b(f(x_0), y_1)$ and $v_1 \geq b(y_1, f(x_2))$,

$$\inf_{x \in f^{-1}(y_1)} (a(x_0, x) \vee v_0) + (a(x, x_2) \vee v_1) \leq a(x_0, x_2) \vee (v_0 + v_1).$$

We recall from Section 2 that, given a non-expansive map $f : (X, a) \to (Y, b)$ and a premetric space (Z, c), the structure d in the partial product P is given by

$$d((s, y), (s', y')) = \min \left\{ v \in \overline{\mathbb{R}}_+ \, ; \begin{array}{l} v \geq b(y, y') \; \& \\ \forall x \in f^{-1}(y) \; \forall x' \in f^{-1}(y') \; a(x, x') \vee v \geq c(s(x), s'(x')) \end{array} \right\},$$

for every $(s, y), (s', y') \in P$.

Theorem. *A non-expansive map $f : (X, a) \to (Y, b)$ between premetric spaces is exponentiable in* **PMet** *if and only if, for each $x_0, x_2 \in X$, $y_1 \in Y$ and $u_0, u_1 \in \overline{\mathbb{R}}_+$ such that $u_0 \geq b(f(x_0), y_1)$, $u_1 \geq b(y_1, f(x_2))$ and $u_0 + u_1 = \max\{a(x_0, x_2), b(f(x_0), y_1) + b(y_1, f(x_2))\} < \infty$,*

$$\forall \varepsilon > 0 \; \exists x_1 \in f^{-1}(y_1) \; : \; a(x_0, x_1) < u_0 + \varepsilon \textit{ and } a(x_1, x_2) < u_1 + \varepsilon. \qquad (**)$$

Proof. First we show that condition $(**)$ is necessary for exponentiability. For u_0, u_1 as above, Theorem 3.4 together with $u_0 + u_1 \geq a(x_0, x_2)$ assure that

$$\inf_{x \in f^{-1}(y_1)} (a(x_0, x) \vee u_0) + (a(x, x_2) \vee u_1) \leq u_0 + u_1,$$

and the conclusion follows.

Conversely, assume that condition $(**)$ is satisfied.

We first observe that, when $\max\{a(x_0, x_2), b(f(x_0), y_1) + b(y_1, f(x_2))\} = \infty$, $(*)$ is immediately verified.

Let $v_0, v_1 \in \overline{\mathbb{R}}_+$ be such that $v_0 \geq b(f(x_0), y_1)$ and $v_1 \geq b(y_1, f(x_2))$.

If $a(x_0, x_2) \leq b(f(x_0), y_1) + b(y_1, f(x_2))$, then necessarily $u_0 = b(f(x_0), y_1)$ and $u_1 = b(y_1, f(x_2))$ in $(**)$. We therefore have

$$\forall \varepsilon > 0 \; \exists x_1 \in f^{-1}(y_1) \; : \; a(x_0, x_1) < u_0 + \varepsilon \leq v_0 + \varepsilon \text{ and } a(x_1, x_2) < u_1 + \varepsilon \leq v_1 + \varepsilon,$$

which implies that

$$\inf_{x \in f^{-1}(y_1)} (a(x_0, x) \vee v_0) + (a(x, x_2) \vee v_1) = v_0 + v_1 = a(x_0, x_2) \vee (v_0 + v_1).$$

Consider now the case $a(x_0, x_2) \geq b(f(x_0), y_1) + b(y_1, f(x_2))$. If $v_0 + v_1 \geq a(x_0, x_2)$, using $(**)$ for $u_0 \leq v_0$ and $u_1 \leq v_1$ such that $u_0 + u_1 = a(x_0, x_2)$, we conclude that

$$\inf_{x \in f^{-1}(y_1)} (a(x_0, x) \vee v_0) + (a(x, x_2) \vee v_1) = v_0 + v_1 = a(x_0, x_2) \vee (v_0 + v_1).$$

If $v_0 + v_1 < a(x_0, x_2)$, let $u_0 = v_0$ and $u_1 = a(x_0, x_2) - v_0$ in $(**)$. Then

$$\begin{aligned}\inf_{x \in f^{-1}(y_1)} (a(x_0, x) \vee v_0) + (a(x, x_2) \vee v_1) &\leq \inf_{x \in f^{-1}(y_1)} (a(x_0, x) \vee u_0) + (a(x, x_2) \vee u_1) \\ &\leq a(x_0, x_2) = a(x_0, x_2) \vee (v_0 + v_1).\end{aligned}$$

□

We observe that, defining, for a premetric space (X, a) and $X' \subseteq X$,

$$\begin{aligned}\downarrow S &:= \{x \in X \,;\, \exists x' \in X \;:\; d(x', x) < \infty\} \text{ and} \\ \uparrow S &:= \{x \in X \,;\, \exists x' \in X \;:\; d(x, x') < \infty\},\end{aligned}$$

one has:

Corollary A. *An embedding $f : (X, a) \to (Y, b)$ in* **PMet** *is exponentiable if and only if*

$$\downarrow f(X) \cap \uparrow f(X) = f(X).$$

□

From the Theorem one derives also the following characterization of exponentiable objects in **PMet**:

Corollary B. *A premetric space (X, a) is exponentiable in* **PMet** *if and only if, for each $x_0, x_2 \in X$ and $u_0, u_1 \in \overline{\mathbb{R}}_+$ such that $u_0 + u_1 = a(x_0, x_2)$,*

$$\forall \varepsilon > 0 \; \exists x_1 \in X \;:\; a(x_0, x_1) < u_0 + \varepsilon \text{ and } a(x_1, x_2) < u_1 + \varepsilon.$$

□

Using this characterization it is obvious that:

- $\mathbb{Q}^+$ with the hom structure (inherited from $\overline{\mathbb{R}}_+$), and $\mathbb{Q}$ and $\mathbb{Q}^+$ with the usual Euclidean metric are exponentiable in **PMet**;
- finite premetric spaces with non-trivial premetric (i.e. having points whose distance differs from 0 and ∞) are not exponentiable in **PMet**.

These examples give an interesting contrast with the situation in **Top**: with their induced topology the former ones are not exponentiable in **Top** while the latter ones are.

4.3 Exponentiation in Met. We finally study the situation in the category **Met** of (symmetric, separated, with non-infinite distances) metric spaces and non-expansive maps.

Proposition. *Let (X, a), (Y, b) and (Z, c) be metric spaces and let $f : (X, a) \to (Y, b)$ be a non-expansive map.*

(a) *If there exists the partial product (P, d) of (Z, c) over f in* **PMet**, *then the premetric d is symmetric and separated.*

(b) *If the partial products of (Z, c) over f exist in both* **PMet** *and* **Met**, *then they coincide.*

Proof. (a) From the description of d above, it is obvious that symmetry of d is inherited by the symmetry of a, b and c. Moreover, $d((s,y),(s',y'))=0$ means that $b(y,y')=0$, hence $y=y'$, and, for every $x,x'\in f^{-1}(y)$, $a(x,x')\geq c(s(x),s'(x'))$. In particular, for $x=x'$ it follows that $c(s(x),s'(x))=0$, hence $s=s'$.

(b) Let (P,d) and (P',d') be the partial products of (Z,c) over f in **PMet** and **Met**, respectively. To show that they coincide, it is enough to show that d cannot take the value ∞. By the universal property of (P,d), there exists a non-expansive bijection $t:(P',d')\to(P,d)$. Hence, $d'\geq d$ and the result follows. $\square$

Theorem. *A non-expansive map $f:(X,a)\to(Y,b)$ between metric spaces is exponentiable in* **Met** *if and only if it satisfies condition $(**)$ and has bounded fibres.*

Proof. We first show that boundedness of the fibres of f, together with $(**)$, is sufficient for exponentiability.

By the proposition above, it remains to be shown that d does not take the value ∞. For each $(s,y),(s',y')\in P$, boundedness of $f^{-1}(y)$ and $f^{-1}(y')$ guarantees that $s(f^{-1}(y))$ and $s'(f^{-1}(y'))$ are bounded subsets of (Z,c), hence $\{c(s(x),s'(x'))\,;\,x\in f^{-1}(y),\,x'\in f^{-1}(y')\}$ is bounded as well; let $u(s,s')$ be its supremum. Then $d((s,y),(s',y'))\leq b(y,y')\vee u(s,s')$, hence d is a metric.

To show the reverse implication, we check first that boundedness of the fibres is necessary for exponentiability of f. Let f be exponentiable. For any $y\in Y$, consider $(Z,c)=(f^{-1}(y),a)$, the identity $s:(Z,c)\to(Z,c)$ and a constant map $s':(Z,c)\to(Z,c)$, taking every point x into a fixed $x_0\in f^{-1}(y)$. Then

$$\begin{aligned}\infty>d((s,y),(s',y)) &\geq \min\{v\,;\,v\geq b(y,y)\ \&\ \forall x\in f^{-1}(y)\ a(x,x)\vee v\geq c(s(x),s'(x))\}\\ &= \min\{v\,;\,\forall x\in f^{-1}(y)\ v\geq a(x,x_0)\},\end{aligned}$$

hence $f^{-1}(y)$ is a bounded subset of X.

It remains to be shown that condition $(**)$ is necessary for exponentiability. First we observe that $(**)$ is trivially verified when $X=\emptyset$. For $X\neq\emptyset$, we first check that, if f is exponentiable in **Met**, then f is surjective. Assume that $y_1\in Y\setminus f(X)$. Consider $x_0\in X$, $y_0=f(x_0)$, and

$$v=\sup\{a(x,x')\,;\,x,x'\in f^{-1}(y_0)\}.$$

Then $v<\infty$ because f has bounded fibres. Let $Z:=\{0,1\}$, with $c(0,1)=\max\{v,b(y_0,y_1)+b(y_1,y_0)\}+1$. Define the constant maps $s:f^{-1}(y_0)\to Z$, $s'':f^{-1}(y_0)\to Z$, which send every x to 0 and 1, respectively, and let $s':f^{-1}(y_1)=\emptyset\to Z$. Then it is easy to check that

$$\begin{aligned}d((s,y_0),(s',y_1)) &= b(y_0,y_1),\\ d((s',y_1),(s'',y_0)) &= b(y_1,y_0),\\ d((s,y_0),(s'',y_0)) &= c(0,1),\end{aligned}$$

which shows that d is not transitive, since

$$d((s,y_0),(s'',y_0))=c(0,1)>b(y_0,y_1)+b(y_1,y_0)=d((s,y_0),(s',y_1))+d((s',y_1),(s'',y_0)).$$

Finally the result follows from an easy adaptation of the argumentation of the proof of Theorem 3.4, using the maps s_0,s_1,s_2, replacing the hom structure in $\overline{\mathbb{R}}_+$ by the usual metric structure in $[0,\infty[$. Continuity of these maps follows from the continuity of s_0, s_1, s_2 of Theorem 3.4 since the metric structure is just the symmetrization of the hom structure of $\overline{\mathbb{R}}_+$. The only detail to check is that, in the definition of s_2,

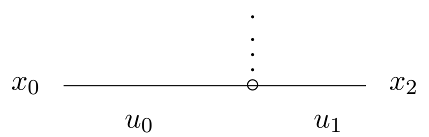
$$\begin{array}{rcl} s_2 : f^{-1}(y_2) & \rightarrow & [0,\infty[\\ x & \mapsto & \inf\limits_{x_1 \in f^{-1}(y_1)} (a(x_0,x_1) \vee v_0) + (a(x_1,x) \vee v_1), \end{array}$$

the meet is non-empty, but this follows immediately from the surjectivity of f. □

In the particular case of metric spaces, we conclude directly the following

Corollary A. *A metric space (X,a) is exponentiable in* **Met** *if and only if it is bounded and, for each $x_0, x_2 \in X$ and $u_0, u_1 \in \overline{\mathbb{R}}_+$ such that $u_0 + u_1 = a(x_2, x_0)$:*

$$\forall \varepsilon > 0 \; \exists x_1 \in X \; : \; a(x_0,x_1) < u_0 + \varepsilon \textit{ and } a(x_1,x_2) < u_1 + \varepsilon :$$



□

Metric spaces satisfying the condition above have been studied in different contexts: in [1], as *almost* 3-*hyperconvex* metric spaces; in [2], under the name *almost convex* spaces, where this condition is used in the study of topologies in the hyperspace of closed subsets of X. In fact, exponentiability of f is equivalent to the "composition of" (closed) balls (see [2], Proposition 4.1.4), as stated below. We denote by $S_u(x_0)$ ($\overline{S}_u(x_0)$) the open (closed) ball with center x_0 and radius u.

Corollary B. *For a bounded metric space (X,a), the following conditions are equivalent:*

(i) (X,a) *is exponentiable;*

(ii) $\forall x_0 \in X \;\; \forall u, v \in]0,\infty[\;\; S_u(S_v(x_0)) = S_{u+v}(x_0)$;

(iii) $\forall x_0 \in X \;\; \forall u, v \in]0,\infty[\;\; \overline{S}_u(\overline{S}_v(x_0)) = \overline{S}_{u+v}(x_0)$. □

Finally, from the Theorem we obtain an interesting characterization of exponentiable complete metric spaces.

Corollary C. *For a complete metric space (X,a), the following conditions are equivalent:*

(i) (X,a) *is exponentiable;*

(ii) (X,a) *is bounded and totally convex, i.e.*
$\forall x_0, x_2 \in X \;\; \forall u_0, u_1 \in [0,\infty[\;\; u_0 + u_1 = a(x_0,x_2) \Rightarrow \exists x_1 \in X \; : \; a(x_0,x_1) = u_0 \;\&\; a(x_1,x_2) = u_1$.

□

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