Affine sets: the structure of complete objects and duality

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Abstract

An existence theorem for completions of categories of T_0 objects of some kind of topological categories over **Set** is given, and an internal characterization of complete objects in these categories is established. As a consequence, we recover the existence of completions in several categories studied in topology (such us closure spaces, α-spaces, topological spaces, approach spaces and fuzzy spaces) together with descriptions of their complete objects. A duality theorem is also provided, rendering many familiar dualities (e.g., Stone duality, Tarski duality) "internal" dualities.

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1 Introduction

The notion of affine set over an algebraic theory was first introduced by Diers ([13, 14]), motivated by his background in algebraic geometry. Roughly speaking, the category of affine sets is a topological category over **Set** having some particularly nice properties. Even though this restricts the rather general notion of a topological category, many familiar categories of topology actually share these additional properties, making therefore this notion very attractive to topologists. Over the last decade various authors developed topological aspects of categories of affine objects of a given complete category, see for instance [8, 9, 10, 11, 12, 18, 20, 21, 22, 23, 19, 25].

Inspired by the Cauchy-completion of a metric space, [4] develops a general theory of completion in a complete category. It is shown that each category admits at most one subcategory of "complete objects", moreover, [4] provides conditions which ensure the existence of such a subcategory. The category of affine sets turned out to be a perfect place to apply the results of [4], which is the topic of this paper. We prove the existence of completions in the category of all separated affine sets, and provide an internal characterization of their complete objects. Here the *Deus ex machina* for the Existence and the Structure theorems is a closure operator in the sense of Dikranjan and Giuli ([16]), called *Zariski closure*, with a number of additional useful properties.

Let A be a given set with some algebraic structure specified by an (infinitary) signature Ω; an *affine set* over A is a pair $(X,\mathcal{U})$ where X is a set and $\mathcal{U}$ is an Ω-subalgebra of the algebra A^X of all functions from the set X to the set A. $(X,\mathcal{U})$ is called *separated* or T_0 whenever $\mathcal{U}$ separates the points of X. An *affine map* from $(X,\mathcal{U})$ to $(Y,\mathcal{V})$ is a function $f : X \to Y$ such that $v \circ f \in \mathcal{U}$ whenever $v \in \mathcal{V}$. $\mathsf{ASet}(\Omega)$ and $\mathsf{ASet}_0(\Omega)$ denote the category of all affine sets and affine maps and the corresponding subcategory of all separated affine sets, respectively.

The main results of this paper are collected in the theorems below:

Existence Theorem (see Theorem 3.2). $\mathsf{ASet}_0(\Omega)$ admits a unique firm reflective subcategory. That is: there exists a subcategory R of $\mathsf{ASet}_0(\Omega)$ satisfying the following conditions: for each X in $\mathsf{ASet}_0(\Omega)$, there exists a z-dense embedding $r_X : X \to RX$ with $RX \in \mathsf{R}$ such that, for every affine map $f : X \to Y$, there exists a unique affine map $f' : RX \to Y$ such that $f' \circ r_X = f$; moreover, f' is an isomorphism whenever f is a z-dense embedding.

The objects of R are called *complete affine sets.* A *one-point extension* of an affine set $(X,\mathcal{U})$ is an affine set $(Y,\mathcal{V})$ with $Y = X+1$ containing $(X,\mathcal{U})$ as an affine z-dense subset via the inclusion map $X \hookrightarrow X+1$.

Structure Theorem (see Theorem 3.6 and Corollary 3.7). The following assertions are equivalent, for a separated affine set $(X,\mathcal{U})$ in $\mathsf{ASet}(\Omega)$.

(i) $(X,\mathcal{U})$ is complete.

(ii) $(X,\mathcal{U})$ admits only non-separated one-point extensions in $\mathsf{ASet}(\Omega)$.

(iii) For every Ω-homomorphism $s : \mathcal{U} \to A$, there exists some $x \in X$ with $u(x) = s(u)$, for all $u \in \mathcal{U}$.

Duality Theorem (see Theorem 3.8). ASet_0 is a self-dual category.

The Duality Theorem follows from the fact that Set is a symmetric monoidal closed category. In fact, the duality theorem is valid for the T_0 affine objects of every complete symmetric monoidal closed category ([25]).

These results apply to a wide variety of topological categories as we show in Section 4. Among these are the categories of T_0-objects of: closure spaces, fields of sets, σ-spaces, topological spaces, P_α-spaces, measurable spaces, spaces of countable tightness, approach spaces, fuzzy topological spaces and the categories of partially ordered sets and of metric spaces. The corresponding complete objects are: the T_0 sober closure spaces, the Stone spaces, the T_0 sober σ-spaces, the T_0 sober topological spaces, the T_0 sober approach spaces (see [3, 19]), the T_0 sober fuzzy spaces (see [31]) and the complete metric spaces (every partially ordered set is complete). Some of the above categories occur in familiar dualities. By the Duality Theorem their duals can be seen as being obtained by formal transposition in $\mathbf{ASet_0}$.

Affine sets over a set A coincide with the normal Chu spaces over A, used by Pratt [30] as a generalization of Nielsen, Plotkin and Winskel's notion of event structure for modeling concurrent computation.

The categorical terminology is that of [1]. For the categorical theory of closure operators we refer to [17]. Throughout this paper, all subcategories are assumed to be full and isomorphism closed.

2 Affine sets

In this section we introduce the basic facts about categories of affine sets over an algebraic theory, and refer for further details to [13, 14].

Definition 2.1. Let Ω be a signature (with arity $a : \Omega \to \mathsf{Card}$, where Card denotes the class of all cardinals) and A be an Ω-algebra.

(a) An *affine set* over a A is a pair $(X, \mathcal{U})$ where X is a set and $\mathcal{U}$ a Ω-subalgebra of A^X.

(b) An *affine map* from $(X, \mathcal{U})$ to $(Y, \mathcal{V})$ is a function $f : X \to Y$ such that $v \circ f \in \mathcal{U}$, for all $v \in \mathcal{V}$.

(c) $(X, \mathcal{U})$ is called *separated* or T_0 if $\mathcal{U}$ separates the points of X.

The category of affine sets over A and affine maps is denoted by $\mathbf{ASet}(\Omega)$, the corresponding subcategory of separated affine sets is denoted by $\mathbf{ASet}_0(\Omega)$. In particular we consider $\Omega = \varnothing$, here an Ω-algebra is just a set and any function is an Ω-homomorphism. Furthermore, an affine set over the set A is a pair $(X, \mathcal{U})$ where $\mathcal{U}$ is just a subset of A^X. In this situation we write simply $\mathbf{ASet}$ respectively $\mathbf{ASet}_0$ instead of $\mathbf{ASet}(\Omega)$ respectively $\mathbf{ASet}_0(\Omega)$.

Proposition 2.2. $\mathbf{ASet}(\Omega)$ *is topological over* $\mathbf{Set}$.

Proof. Let X be a set, $(Y_i, \mathcal{V}_i)_{i \in I}$ be a family of affine sets and $(f_i : X \to Y_i)_{i \in I}$ be a family of functions. The affine structure on X defined by

$$\mathcal{U} = \{v_i \circ f_i \mid i \in I, v_i \in \mathcal{V}_i\}$$

is the initial structure in X induced by $(f_i)_{i \in I}$. □

Some well-known consequences of the above proposition are listed below (see, for instance, [1]).

Corollary 2.3. *1.* $\mathbf{ASet}(\Omega)$ *admits uniquely determined final structures.*

2. An affine map $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is initial if and only if, for every $u \in \mathcal{U}$, there exists $v \in \mathcal{V}$ such that $v \circ f = u$, f is a final affine map if and only if $g \in A^X$ is in $\mathcal{V}$ whenever $g \circ f \in \mathcal{U}$.

An initial injective affine map is, as usual, called embedding*, and its domain is called* affine subset *of its codomain. Dually, every surjective final affine map is called* quotient *map.*

More generally, an affine subobject *of $(Y, \mathcal{V})$ is an equivalence class of embeddings with codomain $(Y, \mathcal{V})$, where two embeddings, $m : (X, \mathcal{U}) \to (Y, \mathcal{V})$ and m' :*

$(X',\mathcal{U}') \to (Y,\mathcal{V})$, are equivalent if there is an isomorphism $\alpha : (X,\mathcal{U}) \to (X',\mathcal{U}')$ with $m' \circ \alpha = m$. Clearly, the affine subobjects of $(Y,\mathcal{V})$ are in bijective correspondence with the subsets of Y.

3. **ASet**(Ω) *admits (surjective affine maps, embeddings) as a factorization structure.*

4. **ASet**(Ω) *is a complete and cocomplete category whose limits (in particular products and equalizers) are lifted from* **Set** *by initial and whose colimits are lifted from* **Set** *by final structures.*

   Thus, in particular, the product of a family $(X_i,\mathcal{U}_i)_{i\in I}$ of affine sets has the cartesian product of the X_i as underlying set and the set of all compositions $u_j \circ p_j$, $j \in I$, $u_j \in \mathcal{U}_j$, as affine structure (p_j is the j^{th} projection).

5. *A full and isomorphism closed subcategory of* **ASet**(Ω) *is quotient reflective if and only if it is epireflective (i.e., it is stable under products and affine subsets) and it is stable under refinements of the structure.*

The category **ASet**(Ω) is in general not well fibered even though every set admits only a set of affine structures. In fact, for $\Omega = \varnothing$ and $\#A \geq 2$, the empty set admits two affine structures and every singleton set admits at least four affine structures. This defect can be removed assuming that the affine structure always contains all constant functions, i.e by adding for each element of A a constant to Ω. The category obtained this way from **ASet**(Ω) we denote as **ASET**(Ω).

The category **ASet**(Ω) admits a canonical closure operator, called Zariski-closure, which turns out to be an important tool in the study of affine sets.

Definition 2.4. Let $(X,\mathcal{U})$ be an affine set and let M be a subset of X. The z-closure of M in $(X,\mathcal{U})$ is defined by

$$z_{(X,\mathcal{U})}M = \bigcap \big\{ \mathrm{Eq}(u,v) \mid u,v \in \mathcal{U}, M \subseteq \mathrm{Eq}(u,v) \big\}$$

where $\mathrm{Eq}(u,v)$ denotes the equalizer of u and v.

Clearly, the Zariski closure on **ASet**(Ω) is the restriction of the Zariski closure on **ASet** to **ASet**(Ω). We will write $z_X M$ or simply zM for the z-closure of M when no confusion is possible. The terms z*-dense affine map*, *z-closed embedding*, and *z-closed affine subset* have the usual meaning.

One easily verifies the following properties (cf. [16]).

Proposition 2.5. *For every affine set X , affine subsets $M \subseteq X$, and affine map $f : X \to Y$:*

1. *$M \subseteq z_X M$ (extensiveness);*

2. *$M, N \subseteq X \Rightarrow z_X M \subseteq z_X N$ (monotonicity);*

3. *$f(z_X M) \subseteq z_Y(fM)$ (continuity);*

4. *$z_X(z_X M) = z_X M$ (idempotency) ;*

5. *$M \subseteq Z \subseteq X \Rightarrow z_Z M = (z_X M) \cap Z$ (heredity).*

Remark 2.6. (a) If $\mathcal{U}$ is empty then $z_X M = X$ for every subset M of X. Consequently the Zariski closure operator is in general not grounded (i.e., does not have the property $z_X \varnothing = \varnothing$ for every affine set X).

(b) The Zariski closure operator is in general not additive (i.e., there exist an affine set X and $M, N \subseteq X$ such that $z_X(M \cup N) \subsetneq z_X M \cup z_X N$ (see Theorem 3.2 in [21]).

(c) Properties 4. and 5. in the above proposition imply that, every subset of every affine set is z-dense in its z-closure. That is:

$$z_{(z_M X)} M = z_X M.$$

The above property, called *weak heredity* in [16, 17], is crucial for many results of the paper.

(d) Properties 1, 2 and 3 say that the Zariski closure is a closure operator in the sense of Dikranjan and Giuli ([16]), so the categorical theory of closure operators ([16]) applies to z.

Proposition 2.7. *(z-dense maps, z-closed embeddings) is a morphism factorization structure of* $\mathsf{ASet}(\Omega)$.

Proposition 2.8. *For an affine set X over A, the following assertions are equivalent.*

1. *X is separated.*
2. *The diagonal morphism $\Delta_X : X \to X \times X$ is z-closed.*
3. *The equalizer of any pair $f, g : X \to Y$ is a closed embedding.*
4. *For an affine map $f : Y \to Z$ and any affine maps $h, k : Z \to X$, if $hf = kf$, then $h = k$.*
5. *Any initial morphism $f : X \to Y$ is an embedding.*

Proposition 2.9. *In* $\mathsf{ASet}_0(\Omega)$ *the epimorphisms are precisely the z-dense affine maps and the regular (= extremal) monomorphisms are precisely the z-closed embeddings.*

In what follows we denote by $\mathbb{A}$ the separated affine set $(A, \langle 1_A \rangle)$, called *Sierpinski affine set over A*. Here $\langle 1_A \rangle$ denotes the subalgebra of A^A generated by the identity map $1_A : A \to A$.

Lemma 2.10. *A function $f : (X,\mathcal{U}) \to \mathbb{A}$ is affine if and only if $f \in \mathcal{U}$. Hence, $\mathbb{A}$ is initially dense in* $\mathsf{ASet}(\Omega)$*; that is, for each affine set $(X,\mathcal{U})$, the source $(f : (X,\mathcal{U}) \to \mathbb{A})_f$ of all affine maps from $(X,\mathcal{U})$ to $\mathbb{A}$ is initial. Furthermore, $\mathbb{A}$ is injective with respect to initial morphisms in* $\mathsf{ASet}(\Omega)$

Of course, for $\Omega = \varnothing$ we have $\mathbb{A} = (A, \{1_A\})$, and therefore $1_A : \mathbb{A} \to \mathbb{A}$ is the only affine endomap of $\mathbb{A}$. Consequently, any affine map into $\mathbb{A}$ is z-dense.

Theorem 2.11. *Let $(X,\mathcal{U})$ be an affine set.*

(a) The affine map

$$\Phi : X \to \mathbb{A}^{\mathcal{U}}$$

induced by $(f : (X,\mathcal{U}) \to \mathbb{A})_f$ *is initial. If* $\Omega = \varnothing$*, then* Φ *is also z-dense.*

(b) Φ *is an embedding if and only if* $(X,\mathcal{U})$ *is separated.*

Corollary 2.12. *(a)* $\mathbf{ASet}_0(\Omega)$ *is quotient reflective in* $\mathbf{ASet}(\Omega)$ *and it is simply cogenerated by the Sierpinski affine set* $\mathbb{A}$.

(b) Every T_0*-reflection is initial.*

(c) The category $\mathbf{ASET}(\Omega)$ *is a universal topological category in the sense of Marny [29].*

For any signature Ω, the category $\mathbf{ASet}(\Omega)$ is a subcategory of **ASet** in a particularly nice way. Below we give a description of those subcategories of **ASet** which arise as $\mathbf{ASet}(\Omega)$ for some Ω. Following [1], we call a subcategory **C** of a concrete category **A** over **Set** *concretely coreflective* if, for each object A in **A**, there exists an identity-carried coreflection map $r : CA \to A$. If **A** is even topological over **Set**, then **C** is as well topological over **Set** ([1, Theorem 21.35]) and, moreover, finally closed in **A** ([1, Theorem 21.24]). **C** is called *hereditary* in **A** if, for each embedding $A \to B$ in **A**, with B also A belongs to **C**. In the sequel we will need a slightly stronger notion: we call **C** *initially hereditary* if, for each initial morphism $A \to B$ in **A**, with B also A belongs to **C**. We remark that a subcategory **C** of **ASet** is a *topological geometric subcategory* (in the sense of [15]) if and only if **C** is concretely coreflective and initially hereditary in **ASet**. Combining now Theorem 3.3 and Proposition 4.4 of [15] gives us the following facts (which were also observed in [7]).

Theorem 2.13. *1. For each signature* Ω *and* Ω*-algebra* A*,* $\mathbf{ASet}(\Omega)$ *is a topological geometric subcategory of* **ASet**.

2. For each topological geometric subcategory **C** *of* **ASet***, there is some signature* Ω *and some* Ω*-algebra structure on* A *such that* **C** *is concretely isomorphic to* $\mathbf{ASet}(\Omega)$.

Remark 2.14. Let **C** be a concretely coreflective and initially hereditary subcategory of **ASet** with coreflector $C : \mathbf{ASet} \to \mathbf{C}$. According to the proof of Theorem 3.3 of [15], we can chose Ω having $\hom(C\mathbb{A}^I, C\mathbb{A})$ as the set of I-ary operation symbols. Then A is in a canonical way an Ω-algebra, and we obtain the following statements:

- For each affine set $(X,\mathcal{U})$ in **C**, $\mathcal{U}$ is a subalgebra of A^X. Hence $\mathbf{C} \subseteq \mathbf{ASet}(\Omega)$.
- $C\mathbb{A}^I$ (calculated in **C**) coincides with A^I (calculated in $\mathbf{ASet}(\Omega)$). Since each affine set $(X,\mathcal{U})$ admits an initial affine map into a power of $\mathbb{A}$, we have $\mathbf{C} = \mathbf{ASet}(\Omega)$.

On the other hand, a concretely coreflective and hereditary subcategory **C** of **ASet** need not be initially hereditary. In this case, **C** is not of the form $\mathbf{ASet}(\Omega)$ for some Ω-algebra structure on A. This is not a serious problem since we are mainly interested in the subcategory $\mathbf{C}_0$ of T_0-objects of **C**, and with the same argumentation as above we obtain $\mathbf{C}_0 = \mathbf{ASet}_0(\Omega)$. Hence our results apply to any concretely coreflective and hereditary subcategory **C** of **ASet**.

3 Complete separated affine sets

Inspired by the Cauchy-completion of a metric space, [4] develops a general theory of a special kind of completion in a complete category **X**. In this section we show how these results apply to our setting. Recall from [4] that a reflective subcategory **R** of $\mathbf{ASet}_0(\Omega)$ is called *embedding-epireflective* if, for every $X \in \mathbf{ASet}_0(\Omega)$, the reflection map $r_X : X \to RX$ is a z-dense embedding. Furthermore, **R** is said an *embedding-firm* epireflective subcategory (firm subcategory for short) if, for every z-dense embedding $f : X \to Y$ with $Y \in \mathbf{R}$, the extension $f' : RX \to Y$ along $r_X : X \to RX$ is an isomorphism. It is shown in [4, Corollary 1.4] that there exists at most one firm subcategory of $\mathbf{ASet}_0(\Omega)$.

The paradigmatic example of such a situation is given by the category **Met** of all metric spaces and non-expansive maps (that is, which satisfy $d(fx, fy) \leq d(x, y)$). **Met** admits as a firm subcategory the full subcategory of all Cauchy-complete metric spaces. This is why we sometimes say that a category **X** *admits completions* [13] when it admits a firm subcategory **R**. Then the objects of **R** are called the *complete objects* of **X**.

Definition 3.1. (a) A separated affine set I is called *z-injective* (weakly injective in [4]) if it is injective with respect to z-dense embeddings. That is: for every z-dense embedding $m : X \to Y$ and every affine map $f : X \to I$, there exists an affine map $f' : Y \to I$ such that $f' \circ m = f$;

(b) A separated affine set is called *absolutely z-closed* (algebraic in [13, 14]) if it is z-closed in every separated affine set into which it can be embedded.

The Sierpinski affine set $\mathbb{A}$ is injective, hence also z-injective. Furthermore, one easily sees that z-injectivity is stable under products and z-closed affine subsets (the latter follows from the (z-dense maps,z-closed embeddings)-diagonalization property, see also [4]). Therefore each absolutely z-closed separated affine set is z-injective. On other hand, given a z-injective separated affine set X, then each z-dense embedding of X into a separated affine set is an isomorphism, hence X is absolutely z-closed. Note that here we only need X to be z-injective in $\mathbf{ASet}_0(\Omega)$, hence a separated affine set X is z-injective in $\mathbf{ASet}(\Omega)$ if and only if X is z-injective in $\mathbf{ASet}_0(\Omega)$. The subcategory of $\mathbf{ASet}(\Omega)$ of all z-injective separated affine sets we denote as **Inj(z)**.

Theorem 1.6 and Corollary 1.7 of [4] give us now

Theorem 3.2. **Inj(z)** *is firmly reflective in* $\mathbf{ASet}_0(\Omega)$.

For every $(X,\mathcal{U}) \in \mathbf{ASet}_0$, the reflection map $r_X : X \to RX$ is given by the restriction of the canonical affine map $\Phi : (X,\mathcal{U}) \to \mathbb{A}^{\mathcal{U}}$ (see Theorem 2.11) to the z-closure RX of $\Phi(X)$. In virtue of Theorem above, we call the z-injective separated affine sets *complete*. Summing up, we have (see also [14, Theorem 5.2]):

Theorem 3.3. *For a separated affine set* $(X,\mathcal{U})$*, the following assertions are equivalent.*

(a) $(X,\mathcal{U})$ *is complete,*

(b) $(X,\mathcal{U})$ *is z-injective,*

(c) $(X,\mathcal{U})$ *is absolutely z-closed;*

(d) $(X,\mathcal{U})$ *is a z-closed affine subset of a product of copies of the Sierpinski affine set* $\mathbb{A}$*;*

In case $\Omega = \varnothing$*, these assertions are also equivalent to*

(e) *The canonical embedding* $\Phi : X \to \mathbb{A}^{\mathcal{U}}$ *is an isomorphism.*

Corollary 3.4. *If* $\Omega = \varnothing$*, then a set* X *admits a complete affine structure if and only if its cardinality is of the form* A^I *for some set* I*.*

Note that the corollary above is in sharp contrast with the classical case of complete metric spaces in the realm of all metric spaces and non-expansive maps. Trivially, every finite metric space is complete, but for an infinite set A, finite affine sets over A are not complete.

Remark 3.5. Recall from [5] that $X \in \mathbf{ASet}(\Omega)$ is z-compact if, for every $Y \in \mathbf{ASet}(\Omega)$, the projection

$$p_2 : X \times Y \to Y$$

sends z-closed affine subsets in z-closed affine subsets (p_2 is z-preserving in the terminology of [5]). It is shown in [23] in a more general context that every z-compact affine set in $\mathbf{ASet}_0(\Omega)$ is also complete in $\mathbf{ASet}_0(\Omega)$.

The converse statement is in general false, see Example (1) of Section 6 in [23]. Veerle Claes and Eva Colebunders [8] found sufficient conditions for z-compactness being equivalent to completeness.

Let $(X,\mathcal{U})$ be an affine set. A *one-point extension* of $(X,\mathcal{U})$ is an affine set $(Y,\mathcal{V})$ with $Y = X+1$ containing $(X,\mathcal{U})$ as an affine z-dense subset via the inclusion map $X \hookrightarrow X+1$. By Theorem 3.3, a separated affine set $(X,\mathcal{U})$ is not complete if and only if $(X,\mathcal{U})$ is a non z-closed affine subset of some separated affine set $(Y,\mathcal{V})$. Clearly, $(Y,\mathcal{V})$ can be chosen as a one-point extension of $(X,\mathcal{U})$, and we conclude:

Theorem 3.6. *A separated affine set* $(X,\mathcal{U})$ *is complete if and only if* $(X,\mathcal{U})$ *admits only non-separated one-point extensions in* $\mathbf{ASet}(\Omega)$*.*

By definition, an affine map $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is initial if and only if the "composition-with-f-map" $\alpha : \mathcal{V} \to \mathcal{U}$ is surjective, f is z-dense if and only if α is injective. In particular, if $(Y,\mathcal{V})$ is a one-point-extension of $(X,\mathcal{U})$, each $u \in \mathcal{U}$ has a unique extension to a map $Y \to A$ in $\mathcal{V}$. Therefore the set $\mathcal{V} \subseteq A^Y$ is uniquely determined by a map $s : \mathcal{U} \to A$ which specifies the values at the additional point $\star \in Y - X$. Given such a map $s : \mathcal{U} \to A$, we define $\mathcal{V}$ to be the image of $\langle i_{\mathcal{U}}, s\rangle : \mathcal{U} \to A^X \times A \cong A^{X+1}$; moreover, $\mathcal{V}$ is a subalgebra of A^Y if and only if $s : \mathcal{U} \to A$ is an Ω-homomorphism.

Corollary 3.7. *Let* $(X,\mathcal{U})$ *be an affine set in* $\mathbf{ASet}(\Omega)$*. Then the folowing assertions hold.*

1. *The one-point-extensions of* $(X,\mathcal{U})$ *in* $\mathbf{ASet}(\Omega)$ *are in one-to-one correspondence with the* Ω*-homomorphisms* $\mathcal{U} \to A$*.*

2. *Let* $(Y,\mathcal{V})$ *be a one-point extensions of* $(X,\mathcal{U})$ *in* $\mathbf{ASet}(\Omega)$*. Then* $(Y,\mathcal{V})$ *is separated if and only if, for all* $x \in X$*,* $\mathrm{ev}_\star \neq \mathrm{ev}_x$*. (Here* ev_y *denotes the evaluation map from* $\mathcal{V}$ *to* A *sending* v *to* $v(y)$*.)*

3. *Let* $(X,\mathcal{U})$ *be separated. Then* $(X,\mathcal{U})$ *is complete in* $\mathsf{ASet}_0(\Omega)$ *if and only if, for each* Ω*-homomorphism* $s:\mathcal{U}\to A$*, there exists some* $x\in X$ *with* $\mathrm{ev}_x=s$.

For every $(X,\mathcal{U})\in\mathsf{ASet}_0$ let $(X,\mathcal{U})^*=(U,\mathcal{X})$ be the affine set where $U=\mathcal{U}$ and $\mathcal{X}$ consists of all the evaluation maps $\mathrm{ev}_x:U\to A$, $x\in X$. Then $(X,\mathcal{U})^*$ is separated and, for every affine map $f:(X,\mathcal{U})\to(Y,\mathcal{V})$, the map $f^*:\mathcal{V}\to\mathcal{U}$ defined by $f^*(v)=v\circ f$ is an affine map between $(Y,\mathcal{V})^*$ and $(X,\mathcal{U})^*$.

Theorem 3.8. *The functor* $()^*$ *is an equivalence rendering* ASet_0 *a self-dual category.*

4 Examples

4.1 Boolean examples

A wide range of interesting topological categories can be described using the two-element set $A=S=\{0,1\}$. The category of affine sets over S can be alternatively described as the category with objects pairs $(X,\mathcal{U})$ where $\mathcal{U}$ is a subset of the powerset PX of X (via the inverse images of 1), an affine map from $(X,\mathcal{U})$ to $(Y,\mathcal{V})$ is a function $f:X\to Y$ such that $f^{-1}V\in\mathcal{U}$ for every $V\in\mathcal{V}$. If S comes with an algebraic structure with respect to a signature Ω, the algebra structure on S^X can be transfered via $PX\cong S^X$ to the powerset PX, and an affine set over S can be viewed as a pair $(X,\mathcal{U})$ where $\mathcal{U}$ is a subalgebra of PX. In this language, $(X,\mathcal{U})$ is separated if and only if, for points x and y in X with $x\neq y$, there exists some $U\in\mathcal{U}$ containing exactly one of these points. For any $x\in X$ we put $\mathcal{U}_x=\{U\in\mathcal{U}\mid x\in U\}$, that is, $\mathcal{U}_x$ is the set of all "open" neighborhoods of x. More general, for a subset F of X we let $\mathcal{U}_F$ denote set of all $U\in\mathcal{U}$ such that $U\cap F\neq\varnothing$.

Of course, any map $s:\mathcal{U}\to S$ can be identified with a subset $a\subseteq\mathcal{U}$. We call a subset $a\subseteq\mathcal{U}$ an Ω-subset of $\mathcal{U}$ if its characteristic map $\chi_a:\mathcal{U}\to S$ is an Ω-homomorphism. By Corollary 3.7, a separated affine set $(X,\mathcal{U})$ is complete in $\mathsf{ASet}_0(\Omega)$ if and only if each Ω-subset $a\subseteq\mathcal{U}$ of $\mathcal{U}$ is of the form $\mathcal{U}_x$, for some $x\in X$.

(1) A separated affine set $(X,\mathcal{U})$ in SSet is complete if and only if, for every $a\subseteq\mathcal{U}$, there exists (a necessarily unique, by separation) $x\in X$ such that $a=\mathcal{U}_x$. Thus a set X admits a complete structure if and only if it is of cardinality 2^I for some set I. In particular not all finite sets admit a complete structure.

The category ASet_0 contains Set as fully embedded subcategory if we just regard as a set every affine set of the form $(X,2^X)$. On the other hand, by the above characterization, every complete affine set $(X,\mathcal{U})$ can be seen to be of the form $(2^{\mathcal{U}},\mathcal{U})$ so with its dual a set. Thus the dual by transposition of the category of complete affine sets is the category of sets.

(2) We consider now $\mathsf{SSET}=\mathsf{SSet}(0,1)$, that is, SSET is the subcategory of SSet with objects all $(X,\mathcal{U})$ where $\varnothing\in\mathcal{U}$ and $X\in\mathcal{U}$. An affine set $(X,\mathcal{U})$ in SSET_0 is complete if and only if $a=\mathcal{U}_x$ whenever $X\in a$ and $\varnothing\notin a$.

(3) We consider $\mathsf{SSET}(\cap)$, that is, the category of all affine sets $(X,\mathcal{U})$ over S where $\mathcal{U}$ is stable under finite intersections. Then $(X,\mathcal{U})$ is $\mathsf{SSET}_0(\cap)$-complete if and only if every proper filter a of $\mathcal{U}$ is of the form $\mathcal{U}_x$.

(4) We consider $\mathbf{SSET}(\cup)$, that is, the category of all affine sets $(X,\mathcal{U})$ over S where $\mathcal{U}$ is stable under finite unions. Then $(X,\mathcal{U})$ is $\mathbf{SSET}_0(\cup)$-complete if and only if every proper ideal a of $\mathcal{U}$ is of the form $\mathcal{U}_x$.

(5) We consider $\mathbf{SSET}(\bigcup)$, that is, the category of all affine sets $(X,\mathcal{U})$ over S where $\mathcal{U}$ is stable under arbitrary unions. These affine sets are also known as *closure spaces* [11]. A T_0 closure space $(X,\mathcal{U})$ is complete if and only if $a = \mathcal{U}_x$ whenever $a \subseteq \mathcal{U}$ is a proper upset and $(\mathcal{U} - a)$ is stable under arbitrary unions, if and only if $a_F = \mathcal{U}_x$ for all nonempty closed subsets F of $(X,\mathcal{U})$, if and only if every nonempty closed set is the closure of a point. These closure spaces are known as T_0 *sober closure spaces.*

(6) We consider $\mathbf{SSET}(\cup,\cap)$, that is, the category of all affine sets $(X,\mathcal{U})$ over S where $\mathcal{U}$ is stable under finite unions and inite intersections. An affine set $(X,\mathcal{U}) \in \mathbf{SSET}_0(\cup,\cap)$ is $\mathbf{SSET}_0(\cup,\cap)$-complete if and only if $a = \mathcal{U}_x$ for every prime filter $a \subseteq \mathcal{U}$.

(7) We consider $\mathbf{SSET}(\bigcup_c,\cap)$, that is, the category of all affine sets $(X,\mathcal{U})$ over S where $\mathcal{U}$ is stable under countable unions and finite intersections. These affine sets are known as *α-spaces.* A T_0 α-space $(X,\mathcal{U})$ is complete if and only if $a = \mathcal{U}_x$ whenever $a \subseteq \mathcal{U}$ is a σ-complete prime filter.

(8) We consider $\mathbf{SSET}(\bigcup,\cap)$ which coincides with the category $\mathbf{Top}$ of *topological spaces.* A T_0 topological space $(X,\mathcal{U})$ is complete if and only if $a = \mathcal{U}_x$ whenever $a \subseteq \mathcal{U}$ is a filter with $(U - a)$ stable under arbitrary unions, if and only if $a = \mathcal{U}_x$ whenever a is a completely prime filter, if and only if, for every $x \in X$, $\mathcal{U}_x = a_F$ for a nonempty irreducible closed set F, if and only if every nonempty irreducible closed set is the closure of a point. These spaces are known as *sober spaces.*

(9) We consider $\mathbf{SSET}(\complement)$, that is, the category of all affine sets $(X,\mathcal{U})$ over S where $\mathcal{U}$ is stable under complements. An affine set $(X,\mathcal{U}) \in \mathbf{SSET}_0(\complement)$ is complete if and only if $a = \mathcal{U}_x$ for each $a \subseteq \mathcal{U}$ with $X \in a$ and $U \in a \iff \complement U \in (\mathcal{U} - a)$.

(10) We consider $\mathbf{SSET}(\bigcup_c,\complement)$. These affine sets are known as *measurable spaces.* A T_0 measurable space $(X,\mathcal{U})$ is complete if and only if $a = \mathcal{U}_x$ whenever $a \subseteq U$ is a σ-complete ultrafilter.

(11) We consider $\mathbf{SSET}(\cup,\complement)$. These affine sets are known as *field of sets.* A T_0 field of sets $(X,\mathcal{U})$ is complete if and only if $a = \mathcal{U}_x$ for every ultrafilter $a \subseteq \mathcal{U}$.

Let us denote by $\tau_{\mathcal{U}}$ the topology generated by $\mathcal{U}$. Note that an affine set $(X,\mathcal{U})$ is a T_0 field of sets if and only if $\tau_{\mathcal{U}}$ is a zerodimensional Hausdorff topology. A T_0 field of sets $(X,\mathcal{U})$ is complete if and only if $(X,\tau_{\mathcal{U}})$ is a Stone space. Moreover, the compactness condition implies that the correspondence $\mathcal{U} \mapsto \tau_{\mathcal{U}}$ between complete field of sets and Stone spaces is bijective. Finally, since a map between complete field of sets is affine if and only if it is continuous for the corresponding topologies, we obtain:

The category of Stone spaces is fully embedded in the category of T_0 field of sets as the subcategory of complete field of sets.

(12) The category **Tight** of topological spaces of countable tightness is clearly hereditary corflective in (**Top**, hence in) **SSet**.

A T_0 space of countable tightness is complete in $\mathbf{Tight}_0$ if and only if every irreducible closed set F, such that there is a countable subset M of F with $a_F = a_M$, is the closure of a point [24].

Every cocountable, uncountable topological space X has X as unique irreducible closed set and it is not fulfilling the condition above. So it is complete in $\mathbf{Tight}_0$ while it is not sober, hence not complete in $\mathbf{Top}_0$.

(13) Let α be an infinite regular cardinal. A P_α-space is a topological space in which every intersection of less than α open sets is open. A T_0 P_α-space is complete if and only if $a = \mathcal{U}_x$ whenever a is a completely prime α-filter [24].

4.2 Other examples

(14) The category [0,1]**Set** of affine sets over the unit interval [0,1] has the category **Fuz** of fuzzy topological spaces as a concretely coreflective and initially hereditary subcategory. Thus, the category $\mathbf{Fuz}_0$ admits completions. In [31] (but see also [2]) it is shown that the complete fuzzy topological spaces are the so called T_0 sober fuzzy spaces.

(15) The category $[0, \infty]$**Set** has the category **AP** of approach spaces [28] and the category **Met** of pseudo metric spaces and non expansive maps as a concretely coreflective and initially hereditary subcategory. Hence, both the categories of T_0 approach spaces and of metric spaces admit completions. An internal characterization of the complete approach spaces is given in [3], [19], and the complete metric spaces are of course well known. Several other examples can be found in [9].

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