

APPROXIMATION IN CONTINUOUS \mathcal{Q} -CATEGORIES

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ABSTRACT. Our work is a fundamental study of the notion of approximation in \mathcal{Q} -categories and in $(\mathbb{U}, \mathcal{Q})$ -categories. Our exposition is categorical but kept as close to language of domain theory as possible. Consequently, we introduce auxiliary, approximating and Scott-continuous distributors, the way-below distributor, and continuity of \mathcal{Q} - and $(\mathbb{U}, \mathcal{Q})$ -categories. We fully characterize J -continuous \mathcal{Q} -categories (resp. $(\mathbb{U}, \mathcal{Q})$ -categories) among all complete \mathcal{Q} -categories (resp. $(\mathbb{U}, \mathcal{Q})$ -categories) in the same ways as continuous domains are characterized among all dekos. By varying the choice of the quantale \mathcal{Q} and the class J of ideals, and by further allowing the ultrafilter monad to act on the quantale, we obtain a flexible theory of continuity that applies to partial orders, metric spaces and to general topological spaces. We demonstrate on examples that our theory unifies some major approaches to quantitative domain theory.

Keywords: quantitative domain theory, continuous domain, way-below, Scott-continuity, quantale-enriched category, projectivity, distributor, complete distributivity.

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1. INTRODUCTION

Quantitative domain theory. The contrast between the needs of denotational semantics and the modelling power that domain theory can offer became well visible when in the early Eighties de Bakker and Zucker [deBZ82] presented a quantitative model of concurrent processes based on metric spaces. Their work was later further generalized by America and Rutten [AR89] who considered a general problem of solving recursive domain equations in the category of metric spaces.

Since that time much effort has been spent on unification of domain-theoretic and metric approaches to denotational semantics, which in practice meant a search for a class of mathematical structures that can serve as (quantitative) domains of computation. As an early example, Smyth proposed a framework based on quasi-metrics and quasi-uniformities [Smy88]. Both of these quantitative structures differ from their “classical” counterparts by discarding symmetry. However, in Smyth’s opinion, in order to accommodate semantic domains used in computer science, a further reformulation of basic definitions involving limits and completeness was necessary. Consequently, he suggested bicomplete totally bounded quasi-uniform spaces [Smy91] as quantitative domains, in his next paper [Smy94] reworked the definition of completeness (that is named Smyth-completeness since then), and introduced so called topological quasi-uniform spaces, in which the quasi-uniform structure is linked to an auxiliary topology by some additional axioms. Smyth’s insight immediately inspired further studies in this direction [Sün93, Kün93, Sün95, Sün97].

Another important structures that unify partial orders and metric spaces are \mathcal{Q} -continuity spaces introduced by Kopperman [Kop88]. The idea was to use a non-symmetric distance that takes values in a set \mathcal{Q} with a rich order structure. Flagg [F197] suggested that \mathcal{Q} should be a *value quantale*, that is, a completely distributive unital quantale in which the set of elements that are approximated by the unit is a filter. Soon both authors published a joint paper [FK97] summarizing their research.

Since F.W. Lawvere’s famous 1973 paper [Law73] it is well-known that both ordered sets and metric spaces can be viewed as \mathcal{Q} -enriched categories of Eilenberg and Kelly [EK66, Kel82]: the former ones for the quantale $\mathcal{Q} = 2$, the latter ones for the quantale $\mathcal{Q} = [0, \infty]$. Clearly Kopperman and Flagg’s \mathcal{Q} -continuity spaces are exactly categories enriched in a value quantale \mathcal{Q} .

Smyth’s and Lawvere’s ideas have been combined together in a series of papers by the Amsterdam research group at CWI [BvBR96, Rut98, BvBR98] that showed, among other things, how

to construct the (sequential Yoneda) completion and powerdomains for $[0, \infty]$ -categories. Their work has been complemented by Künzi and Schellekens in [KS02] (they proposed the netwise version of the Yoneda completion). A completion by flat modules for gmses (resp. completion by type 1 filters) was further discussed by Vickers [Vic05] (resp. by Schmitt [Sch06]). Independently, Flagg, Sünderhauf and Wagner [FS02, FSW96] studied ideal completion of \mathcal{Q} -continuity spaces and they in effect demonstrated that for $\mathcal{Q} = [0, \infty]$ their results phrased in terms of ideals (called FSW-ideals here) agree with results of the CWI group phrased in terms of Cauchy nets. They also gave a representation theory for *algebraic* \mathcal{Q} -continuity spaces.

Furthermore, in [Wag94] and later in [Wag97], Wagner proposed a framework for solving recursive domain equations in certain complete \mathcal{Q} -categories, thereby unifying original attempts of Scott [GHK+03, SP82] and de Bakker and Zucker [deBZ82] that in the Eighties seemed to be fundamentally different.

In summary, by 1998 it was known how to construct a variety of completions (some of them turned to be equivalent), generalized Scott and Alexandroff topologies, powerdomains, and how to solve recursive domain equations in the \mathcal{Q} -categorical setting. Since then these ideas of domain-theoretic origin have been successfully applied in semantics. Most notably, solving recursive equations over metric spaces proved to be one of the fundamental tools in semantics of concurrency, see e.g. [vBr01, vBW05, vBMOW05, vBHMW07].

Our motivation and related work. But domain theory can offer much more than tools for solving equations. Quoting from [AJ94]:

Continuity is a central pillar of Domain theory. It serves as a qualitative approximation to computability. [...] it is not sufficient to work only with a notion of “completeness” or “convergence”; one also needs a notion of *approximation*, which does justice to the idea that infinite objects are given in some coherent way as limits of their finite approximations. This leads to considering, not arbitrary complete partial orders, but the continuous ones.

Our work is thought as a fundamental study of the notion of approximation in \mathcal{Q} -categories; it continues a line of research sketched in the preceding section. Our exposition is categorical but kept close to domain-theoretic language of [AJ94, GHK+03]. Consequently, we speak about auxiliary, approximating and Scott-continuous \mathcal{Q} -distributors, about the way-below \mathcal{Q} -distributor, and we introduce continuous \mathcal{Q} -categories.

The generalization from domain theory to \mathcal{Q} -categories that we propose proceeds on various levels, as we shall explain below, comparing our paper to related work in the area.

Relative continuity. There is no canonical choice for \mathcal{Q} -categorical counterparts of even the most fundamental notions of domain theory. For instance, as we saw above, order ideals can be generalized to several non-equivalent concepts on the \mathcal{Q} -level (e.g. forward Cauchy nets, flat modules, FSW-ideals) which nevertheless yield the same definitions in both metric and order-theoretic cases [FSW96, BvBR98, Vic05]. Consequently, one obtains different notions of (co)completeness for \mathcal{Q} -categories based on a specific choice of ideals. The starting point of our paper is a conviction that one has not make this choice right at the beginning, and we study cocompleteness and continuity of \mathcal{Q} -categories relative to an abstract class of ideals J subject to suitable axioms. Accordingly, we speak about J -cocompleteness and J -continuity. As far as there are many papers in the literature dealing with relative cocompleteness [AK88, KS05, CH08, LZ07], we are not aware of any systematic study of relative continuity in \mathcal{Q} -categories.

We therefore introduce a concept of a J -continuous \mathcal{Q} -category and develop its basic characterisations. For appropriate choices of \mathcal{Q} and J we recover many of the well-known classical structures: continuous domains, completely distributive complete lattices, Cauchy-complete metric spaces but there remain many more settings where the meaning of J -continuity is still to be explored.

Algebraicity. Algebraic \mathcal{Q} -continuity spaces (resp. algebraic gmses) were introduced in [FSW96] (resp. in [BvBR98]) Although in this paper we do not study algebraic \mathcal{Q} -categories explicitly, it

is easily seen that for the class J of FSW-ideals (see Example 3.5), our concept of J -algebraicity (that one can immediately deduce from our concept of J -continuity) agrees with the one introduced in the literature.

Continuous categories. The difference between continuous categories of Johnstone and Joyal [Joh82, Kos86, ALR03] and our J -continuous \mathcal{Q} -categories is that the former are **Set**-based and their continuity is not relative to the choice of ideals. On the other hand, our Theorem 4.23(i) confirms that in essence we introduce continuity in the same way — by the requirement that the left adjoint to the Yoneda embedding itself has a left adjoint.

Other relevant literature. Firstly, continuity of complete \mathcal{Q} -categories is a subject of [ZF05]. However, [ZF05] assumes that \mathcal{Q} is a completely distributive complete Heyting algebra, and fixes the choice of ideals to the ones that we consider in Example 3.6. Furthermore, our approach seems to be more elementary, since the Authors of [ZF05] propose a rather complicated definition of continuity, with no single counterpart for the way-below relation.

Secondly, in [Stu07] Stubbe considers totally continuous cocomplete \mathcal{Q} -categories enriched over a quantaloid \mathcal{Q} . On the one hand, a significant part of results from [Stu07] can be recovered from our paper as soon as we fix J to be the class of all \mathcal{Q} -distributors (see Example 3.1). On the other hand, Stubbe shows that instead with quantales it is possible to work with more general quantaloids. It should be acknowledged that our Section 4.4 draws ideas from a very interesting discussion of projectivity in [Stu07].

$(\mathbb{U}, \mathcal{Q})$ -categories. In the last part of our paper we propose a further substantial generalization of continuous domains by considering so called $(\mathbb{U}, \mathcal{Q})$ -categories, where the ultrafilter monad \mathbb{U} is allowed to act on the quantale \mathcal{Q} . As an example we note that $(\mathbb{U}, \mathbf{2})$ -categories are precisely topological spaces. In Section 5.7 we introduce J -continuous $(\mathbb{U}, \mathcal{Q})$ -categories and show that defining approximation — while still possible ‘locally’ — becomes difficult globally, which is of course a price paid for such a generous generality. We close the paper by giving a full characterization of J -continuous $(\mathbb{U}, \mathcal{Q})$ -categories among all $(\mathbb{U}, \mathcal{Q})$ -categories in the same ways as continuous domains are characterized among all deposes.

It is worth mentioning that the ultrafilter monad \mathbb{U} is made compatible with the quantale structure \mathcal{Q} by the convergence structure of a compact topology on \mathcal{Q} . Under some natural assumptions this topology happens to be the Lawson topology, and this observation simplifies the presentation of our results.

2. PRELIMINARIES

2.1. Quantales. A $\mathcal{Q} = (Q, \leq, \otimes, \mathbf{1})$ is a completely distributive commutative unital quantale (in short: a quantale) if: (a) (Q, \leq) is a completely distributive complete lattice; (b) \mathcal{Q} is equipped with an associative, commutative operation $\otimes: Q \times Q \rightarrow Q$ called tensor that distributes over arbitrary suprema, i.e. $a \otimes \bigvee S = \bigvee \{a \otimes s \mid s \in S\}$ for all $a \in Q$ and $S \subseteq Q$; (c) \mathcal{Q} has a element $\mathbf{1}$, called unit, that satisfies $a \otimes \mathbf{1} = a$ for all $a \in Q$;

A quantale \mathcal{Q} is a Girard quantale if moreover: (d) \mathcal{Q} has a element \perp , called the dualizing element, that satisfies $a = Q(Q(a, \perp), \perp)$ for all $a \in Q$. Here $Q(-, -)$ is defined as the right adjoint to tensor, i.e.: $a \otimes b \leq c$ if and only if $a \leq Q(b, c)$ holds for all $a, b, c \in Q$; (e) the dualizing element is the least element and the unit element is the top element with respect to the order on Q . We also assume that $\perp \neq \mathbf{1}$. (We use the nomenclature of Yetter [Yet90] who, however, defines Girard quantales without the condition (e) and without the assumption of commutativity.)

Example 2.1. The two element lattice $\{\perp, \mathbf{1}\}$ with \otimes being conjunction is an example of a Girard quantale. It is denoted by $\mathbf{2}$.

Example 2.2. The unit interval $[0, 1]$ in the order opposite to the natural one, with truncated addition as tensor, is a Girard quantale.

Example 2.3. The extended real half line $[0, \infty]$ in the order opposite to the natural one, with addition as tensor, is an example of a quantale which is not a Girard quantale.

2.2. **Q-relations.** The category $\mathcal{Q}\text{-Rel}$ has sets as objects, and a morphism $r : X \multimap Y$ is simply a function $r : X \times Y \rightarrow \mathcal{Q}$; its composite with $s : Y \multimap Z$ is given by

$$(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z).$$

There is a functor $\mathbf{Set} \rightarrow \mathcal{Q}\text{-Rel}$ which maps objects identically and interprets a map $f : X \rightarrow Y$ as a \mathcal{Q} -relation $f : X \multimap Y$:

$$f(x, y) = \begin{cases} \mathbf{1} & \text{if } f(x) = y, \\ \perp & \text{otherwise.} \end{cases}$$

The hom-sets of $\mathcal{Q}\text{-Rel}$ carry the pointwise order of \mathcal{Q} , so that $\mathcal{Q}\text{-Rel}$ becomes an ordered category. In fact, each hom-set of $\mathcal{Q}\text{-Rel}$ is a complete lattice. Consequently, for every $r : X \multimap Y$, composition by r in $\mathcal{Q}\text{-Rel}$ from either side has a right adjoint, given by *extensions* and *liftings* respectively:

$$(-) \cdot r \dashv (-) \bullet r$$

$$r \cdot (-) \dashv r \bullet (-)$$

$$\frac{t \cdot r \leq s}{t \leq s \bullet r}$$

$$\frac{r \cdot t \leq s}{t \leq r \bullet s}$$

$$\begin{array}{ccc} X & & \\ \downarrow r & \searrow s & \\ Y & \xrightarrow{t} & Z \end{array}$$

$$\begin{array}{ccc} Y & & \\ \uparrow r & \swarrow s & \\ X & \xleftarrow{t} & Z \end{array}$$

$$(s \bullet r)(y, z) = \bigwedge_{x \in X} Q(r(x, y), s(x, z)) \quad (r \bullet s)(z, x) = \bigwedge_{y \in Y} Q(r(x, y), s(z, y))$$

$\mathcal{Q}\text{-Rel}$ has a contravariant involution $(\mathcal{Q}\text{-Rel})^{\text{op}} \rightarrow \mathcal{Q}\text{-Rel}$ which maps objects identically and assigns to $r : X \multimap Y$ its opposite relation $r^\circ : Y \multimap X$ defined by $r^\circ(y, x) = r(x, y)$. When applied to a map f , one obtains an adjunction $f \dashv f^\circ$ in the ordered category $\mathcal{Q}\text{-Rel}$.

2.3. **Q-categories.** A \mathcal{Q} -category $X = (X, X(-, -))$ is a set X with a \mathcal{Q} -relation $X : X \multimap X$ satisfying $1_X \leq X$ (reflexivity), and $X \cdot X \leq X$ (transitivity), or, pointwise: $\mathbf{1} \leq X(x, x)$ and $X(x, y) \otimes X(y, z) \leq X(x, z)$ for all $x, y, z \in X$. A \mathcal{Q} -functor $f : X \rightarrow Y$ must satisfy $f \cdot X \leq Y \cdot f$; pointwise: $X(x, y) \leq Y(fx, fy)$ for all $x, y \in X$. The resulting category $\mathcal{Q}\text{-Cat}$ is isomorphic to the category \mathbf{Ord} of (pre)ordered sets if $\mathcal{Q} = \mathbf{2}$, to Lawvere's category \mathbf{Met} of (pre)metric spaces [Law73] if $\mathcal{Q} = [0, \infty]$ or $\mathcal{Q} = [0, 1]$ (metrics bounded by 1 in the latter case). For the trivial quantale one has $\mathbf{1}\text{-Cat} = \mathbf{Set}$. Furthermore, \mathcal{Q} with its internal hom becomes a \mathcal{Q} -category.

$\mathcal{Q}\text{-Cat}$ is a symmetric monoidal closed category, with tensor product $X \otimes Y((x, y), (x', y')) = X(x, x') \otimes Y(y, y')$, and internal hom: $Y^X(f, g) = \bigwedge_{x \in X} Y(fx, gx)$. The internal hom describes the pointwise order if $\mathcal{Q} = \mathbf{2}$, and the non-symmetrized sup-metric if $\mathcal{Q} = [0, \infty]$ or $\mathcal{Q} = [0, 1]$.

2.4. **Q-distributors.** The category $\mathcal{Q}\text{-Dist}$ has \mathcal{Q} -categories as objects and \mathcal{Q} -distributors as morphisms; here a \mathcal{Q} -distributor $\varphi : X \multimap Y$ is a \mathcal{Q} -relation $\varphi : X \multimap Y$ with $\varphi \cdot X \leq \varphi$ and $Y \cdot \varphi \leq \varphi$. Since always $\varphi = \varphi \cdot 1_X \leq \varphi \cdot X$ and $\varphi = 1_Y \cdot \varphi \leq Y \cdot \varphi$, one actually has $\varphi \cdot X = \varphi$ and $Y \cdot \varphi = \varphi$ for a \mathcal{Q} -distributor $\varphi : X \multimap Y$. In particular, the \mathcal{Q} -distributor $X : X \multimap X$ assumes the role of the identity morphism on X in $\mathcal{Q}\text{-Dist}$, and we write $X = 1_X^*$, in order not to confuse it with 1_X in $\mathcal{Q}\text{-Cat}$. This notation is extended to arbitrary maps $f : X \rightarrow Y$ by $f_* := Y \cdot f$ and $f^* := f^\circ \cdot Y$, and one easily verifies:

Lemma 2.4. *The following are equivalent:*

- (1) $f : X \rightarrow Y$ is a \mathcal{Q} -functor;
- (2) f_* is a \mathcal{Q} -distributor $f_* : X \multimap Y$;
- (3) f^* is a \mathcal{Q} -distributor $f^* : Y \multimap X$.

We obtain functors

$$(-)_* : \mathcal{Q}\text{-Cat} \rightarrow \mathcal{Q}\text{-Dist} \quad \text{and} \quad (-)^* : \mathcal{Q}\text{-Cat}^{\text{op}} \rightarrow \mathcal{Q}\text{-Dist},$$

where $X_* = X = X^*$, for each \mathcal{Q} -category X .

$\mathcal{Q}\text{-Dist}$ is actually an ordered category, where \mathcal{Q} -distributors are compared as \mathcal{Q} -relations. Two \mathcal{Q} -distributors $\varphi, \psi : X \multimap Y$ are equal if and only if $\varphi \cdot x_* = \psi \cdot x_*$ for all $x \in X$, if and only if $y^* \cdot \varphi = y^* \cdot \psi$ for all $y \in Y$. Furthermore, suprema, extensions and liftings of \mathcal{Q} -distributors formed in $\mathcal{Q}\text{-Rel}$ are again \mathcal{Q} -distributors. Moreover the adjunction $f \dashv f^\circ$ in $\mathcal{Q}\text{-Rel}$ extends to an adjunction $f_* \dashv f^*$ in $\mathcal{Q}\text{-Dist}$.

We can transport the order-structure on hom-sets from $\mathcal{Q}\text{-Dist}$ to $\mathcal{Q}\text{-Cat}$ via the functor $(-)^* : \mathcal{Q}\text{-Cat}^{\text{op}} \rightarrow \mathcal{Q}\text{-Dist}$, that is, we define $f \leq g$ whenever $f^* \leq g^*$. Clearly, we have $f \leq g$ if and only if $g_* \leq f_*$. With this definition we turn $\mathcal{Q}\text{-Cat}$ into an ordered category, and therefore the (representable) forgetful functor $\mathcal{Q}\text{-Cat} \rightarrow \mathbf{Set}$ factors through $\mathcal{Q}\text{-Cat} \rightarrow \mathbf{Ord}$. Explicitly, the underlying order of a \mathcal{Q} -category X is given by

$$x \leq_X y \iff \mathbf{1} \leq X(x, y).$$

The following result connects distributors with functors.

Theorem 2.5. *The following are equivalent for \mathcal{Q} -relations $\varphi : X \multimap Y$ between \mathcal{Q} -categories.*

- (i) $\varphi : X \multimap Y$ is a \mathcal{Q} -distributor.
- (ii) $\varphi : X^{\text{op}} \otimes Y \rightarrow \mathcal{Q}$ is a \mathcal{Q} -functor.

Example 2.6. For $\mathcal{Q} = \mathbf{2}$, a \mathcal{Q} -distributor from X to $\mathbf{1}$ corresponds to a monotone map from X^{op} to $\mathbf{2}$, that is, to a lower set in X .

Of particular importance is the \mathcal{Q} -category $\widehat{X} := \mathcal{Q}^{X^{\text{op}}}$ whose elements can be seen either as \mathcal{Q} -functors $X^{\text{op}} \rightarrow \mathcal{Q}$ or as \mathcal{Q} -distributors $X \multimap \mathbf{1}$. Here $\widehat{X}(f, g) = \bigwedge_{x \in X} \mathcal{Q}(fx, gx)$, and we obtain at once the following useful fact:

Lemma 2.7. *For \mathcal{Q} -distributors $\varphi : X \multimap Y, \psi : X \multimap Z$,*

$$\psi \bullet \varphi = \widehat{X}(\ulcorner \varphi \urcorner(-), \ulcorner \psi \urcorner(-)).$$

With $\varphi = X = 1_X^* : X \multimap X$ we have in particular the \mathcal{Q} -functor $X : X^{\text{op}} \otimes X \rightarrow \mathcal{Q}$ whose mate $\ulcorner X \urcorner$ is the Yoneda \mathcal{Q} -functor $y : X \rightarrow \widehat{X}, x \mapsto X(-, x)$. From Lemma 2.7 we deduce

Corollary 2.8 (Yoneda Lemma). *For all $x \in X$ and $f \in \widehat{X}$, $\widehat{X}(yx, f) = fx$.*

Corollary 2.9. *For every \mathcal{Q} -distributor $\varphi : X \multimap Y$, $\ulcorner \varphi \urcorner^* \cdot y_* = \varphi$.*

One calls a \mathcal{Q} -functor $f : X \rightarrow Y$ fully faithful if $X(x, z) = Y(fx, fz)$ for all $x, z \in X$; equivalently, if $X = f^* \cdot f_*$, or just $X \geq f^* \cdot f_*$ (since the other inequality comes for free). By the Yoneda Lemma, the Yoneda embedding $y : X \rightarrow \widehat{X}$ is fully faithful.

For a \mathcal{Q} -distributor $\varphi : X \multimap Y$, we have monotone maps

$$(-) \cdot \varphi : \widehat{Y} \rightarrow \widehat{X} \quad \text{and} \quad (-) \bullet \varphi : \widehat{X} \rightarrow \widehat{Y}$$

which define an adjunction $(-) \cdot \varphi \dashv (-) \bullet \varphi$ in \mathbf{Ord} . Furthermore, one easily verifies that both $(-) \cdot \varphi$ and $(-) \bullet \varphi$ are \mathcal{Q} -functors, so that $(-) \cdot \varphi \dashv (-) \bullet \varphi$ in $\mathcal{Q}\text{-Cat}$. Given now a \mathcal{Q} -functor $f : X \rightarrow Y$, we have $f_* \dashv f^*$ in $\mathcal{Q}\text{-Dist}$ and therefore obtain a chain of adjunctions

$$(-) \cdot f^* \dashv (-) \cdot f_* \dashv (-) \bullet f_*$$

$$\begin{array}{ccc} & \xrightarrow{(-) \cdot f^*} & \\ & \perp & \\ \widehat{Y} & \xrightarrow{(-) \cdot f_*} & \widehat{X} \\ & \perp & \\ & \xrightarrow{(-) \bullet f_*} & \end{array}$$

in $\mathcal{Q}\text{-Cat}$. In the sequel we will write \underline{f} instead of $(-) \cdot f^*$, and denote its right adjoint $(-) \cdot f_*$ as \widehat{f} . In fact, the mappings

$$(f : X \rightarrow Y) \mapsto (f^* : Y \multimap X) \quad \text{and} \quad (\varphi : X \multimap Y) \mapsto ((-) \cdot \varphi : \widehat{Y} \rightarrow \widehat{X})$$

define contravariant functors between $\mathcal{Q}\text{-Cat}$ and $\mathcal{Q}\text{-Dist}$ which are indeed part of a dual adjunction where the units are given by the families $(y : X \rightarrow \widehat{X})_X$ and $(y_* : X \multimap \widehat{X})_X$ respectively. The naturality condition for the latter one gives in particular:

Lemma 2.10. *For any \mathcal{Q} -functor $f : X \rightarrow Y$, $\underline{f^*} \cdot y_* = y_* \cdot f^*$.*

As usual, we call a \mathcal{Q} -category X cocomplete if X has all weighted colimits. Recall that a weighted diagram in X is given by a \mathcal{Q} -functor $d : A \rightarrow X$ together with a weight $\psi : A \multimap B$ in $\mathcal{Q}\text{-Dist}$ on A . A colimit $\text{colim}(d, \psi)$ of such a diagram is a \mathcal{Q} -functor $g : B \rightarrow X$ which represents the \mathcal{Q} -distributor $d_* \bullet \psi$ in the sense that $g_* = d_* \bullet \psi$. A \mathcal{Q} -functor $f : X \rightarrow Y$ preserves the φ -weighted colimit of d if $f \cdot \text{colim}(d, \psi) \cong \text{colim}(f \cdot d, \psi)$, that is, if $(f \cdot g)_* = (f \cdot d)_* \bullet \psi$. Furthermore, $f : X \rightarrow Y$ is cocontinuous if f preserves all weighted colimits which exist in X . One easily verifies that for both cocompleteness and cocontinuity it is enough to verify diagrams where $d = 1_X$ and $B = 1$. A colimit of such a diagram we will also call a supremum of ψ . Now, for $\psi \in \widehat{X}$ and $x \in X$, we observe that

$$x \cong \text{colim}(1_X, \psi) \iff X(x, -) = X \bullet \psi \iff X(x, -) = \widehat{X}(\psi, y(-)) \iff x_* = y^* \cdot \psi_*$$

and conclude that X is cocomplete if and only if the Yoneda embedding $y : X \rightarrow \widehat{X}$ has a left adjoint \mathcal{S} . In this case, the colimit of a general diagram given by $d : A \rightarrow X$ and $\psi : A \multimap B$ can be calculated as $\text{colim}(d, \psi) \cong \mathcal{S} \cdot \underline{d} \cdot \ulcorner \psi \urcorner$.

Lemma 2.11. *For \mathcal{Q} -distributors $\varphi : X \multimap Y$ and $\psi : Y \multimap Z$, $\ulcorner \psi \cdot \varphi \urcorner \cong \text{colim}(\ulcorner \varphi \urcorner, \psi)$.*

Corollary 2.12. *Each $\varphi \in \widehat{X}$ is the φ -weighted colimit of $y : X \rightarrow \widehat{X}$.*

For example, if X is a poset, then for any $\varphi \in \widehat{X}$ we have $\varphi = \bigcup_{x \in \varphi} \downarrow x$.

3. J -COCOMPLETE \mathcal{Q} -CATEGORIES

We will now consider a relativised version of cocompleteness and assume that a subcategory J of $\mathcal{Q}\text{-Dist}$ is given and that it contains all \mathcal{Q} -distributors of the form $f^* \in J$ for a \mathcal{Q} -functors f and satisfies

$$(*) \quad \text{if } y^* \cdot \varphi \in J \text{ for all } y \in Y, \text{ then } \varphi \in J.$$

This condition implies that J is fully determined by all \mathcal{Q} -distributors of type $X \multimap 1$. Therefore, we will often only specify a set of \mathcal{Q} -distributors of type $X \multimap 1$ for each \mathcal{Q} -category X , with the understanding that J is generated by this collection of \mathcal{Q} -distributors with help of (*). However, in this case we must check that the resulting J is indeed closed under composition.

We call a \mathcal{Q} -category X J -cocomplete if X has all weighted colimits where the weight belongs to J . As above, it is enough to consider diagrams where $d = 1_X$ and $B = 1$.

Example 3.1. For any \mathcal{Q} , we can choose $J = \mathcal{Q}\text{-Dist}$. Then J -cocomplete means cocomplete.

Example 3.2. For $\mathcal{Q} = \mathbf{2}$, we consider all \mathcal{Q} -distributors of type $X \multimap 1$ corresponding to ideals in X , which indeed generate a subcategory $J = \mathbf{Idl}$ of $\mathcal{Q}\text{-Dist}$. Then an ordered set X is J -cocomplete if and only if each order-ideal on X has a supremum.

Example 3.3. For $\mathcal{Q} = [0, \infty]$ or $\mathcal{Q} = [0, 1]$, we consider all \mathcal{Q} -distributors of type $X \multimap 1$ corresponding to ideals in X in the sense of [BvBR98]. Again, they generate a subcategory of $\mathcal{Q}\text{-Dist}$, which we denote as \mathbf{FC} . These ideals correspond to equivalence classes of forward Cauchy sequences on X . Hence, X is J -cocomplete if and only if each forward Cauchy sequence on X converges.

Example 3.4. For any \mathcal{Q} we can choose J to be the subcategory of $\mathcal{Q}\text{-Dist}$ consisting of all right adjoint \mathcal{Q} -distributors. Recall from [Law73] that, for $\mathcal{Q} = [0, \infty]$ and for $\mathcal{Q} = [0, 1]$, a right adjoint \mathcal{Q} -distributor $X \dashv \rightarrow 1$ corresponds to an equivalence class of Cauchy sequences on X . A metric space X is J -cocomplete if each Cauchy sequence on X converges.

Example 3.5. For any completely distributive quantale \mathcal{Q} and any \mathcal{Q} -category X , a \mathcal{Q} -distributor $\psi: X \dashv \rightarrow 1$ is a FSW-ideal if:

- (1) $\bigvee_{z \in X} \psi(z) = \mathbf{1}$, and
- (2) for all $\varepsilon_1, \varepsilon_2, \delta \prec \mathbf{1}$, for all $x_1, x_2 \in X$, whenever $\varepsilon_1 \prec \psi(x_1)$ and $\varepsilon_2 \prec \psi(x_2)$, then there exists $z \in X$ such that $\delta \prec \psi(z)$, $\varepsilon_1 \prec \psi(x_1) \otimes X(x_1, z)$ and $\varepsilon_2 \prec \psi(x_2) \otimes X(x_2, z)$.

One easily checks that the generated class $J = \mathbf{FSW}$ is indeed a subcategory of $\mathcal{Q}\text{-Dist}$. Note that for $\mathcal{Q} = [0, \infty]$ and for $\mathcal{Q} = [0, 1]$ FSW-ideals on X are in a bijective correspondence with equivalence classes of forward Cauchy nets on X [FSW96]; for $\mathcal{Q} = \mathbf{2}$, FSW-ideals are characteristic maps of order-ideals on X . Therefore this example unifies Examples 3.2, 3.3.

Example 3.6. For any \mathcal{Q} we can choose J to be the subcategory of $\mathcal{Q}\text{-Dist}$ consisting of all \mathcal{Q} -distributors of type $X \dashv \rightarrow 1$ that satisfy:

- (1) $\bigvee_x \phi(x) = \mathbf{1}$, and
- (2) $\phi(x) \otimes \phi(y) \leq \bigvee_z (\phi(z) \otimes X(x, z) \otimes X(y, z))$.

This definition appears in [Win01] and is also used in [ZF05].

Further examples are mentioned in [Sch06, CH08].

For a \mathcal{Q} -category X , we put

$$J(X) = \{\psi: X \dashv \rightarrow 1 \mid \psi \in J\}$$

and

$$J_S(X) = \{\psi \in J(X) \mid \psi \text{ has a supremum in } X\},$$

both considered as a \mathcal{Q} -subcategory of \widehat{X} . For any $x \in X$, $x^* \in J(X)$, hence the Yoneda functor $y(x) = x^*$ is of type $X \rightarrow J(X)$. Since $y^* \cdot (yx)_* = y^* \cdot y_* \cdot x_* = x_*$ for any $x \in X$, we can further corestrict the Yoneda functor and obtain a \mathcal{Q} -functor $y: X \rightarrow J_S(X)$. On the other hand, there is a map $\mathcal{S}: J_S(X) \rightarrow X$ which to each $\psi \in J_S(X)$ assigns its supremum. It is in fact a \mathcal{Q} -functor. Clearly then, $y^* \cdot \psi_* = (\mathcal{S}(\psi))_* = \mathcal{S}_* \cdot \psi_*$ for any $\psi \in J_S(X)$, i.e. $y^* = \mathcal{S}_*$, which is equivalent to say that \mathcal{S} is a left adjoint to the restriction $y: X \rightarrow J_S(X)$ of the Yoneda \mathcal{Q} -functor: $\mathcal{S} \dashv y$.

Definition 3.7. A \mathcal{Q} -functor $f: X \rightarrow Y$ is *Scott-continuous* if for all $\varphi \in J_S(X)$, $f(\mathcal{S}\varphi) \cong \mathcal{S}f(\varphi)$.

Equivalently, f is Scott-continuous if and only if f preserves all colimits with weight in J which exist in X . Clearly, every cocontinuous \mathcal{Q} -functor is Scott-continuous. For $\mathcal{Q} = \mathbf{2}$ and $J = \mathbf{Idl}$ Scott-continuity has the usual meaning. For $\mathcal{Q} = [0, 1]$ or $\mathcal{Q} = [0, \infty]$ and $J = \mathbf{FC}$, f is Scott-continuous if and only if f preserves limits of forward Cauchy sequences.

Proposition 3.8. *Let Y be a J -cocomplete category. Then any \mathcal{Q} -functor $f: X \rightarrow Y$ uniquely extends to a Scott-continuous \mathcal{Q} -functor $F: J(X) \rightarrow Y$.*

Theorem 3.9. *The inclusion functor $J\text{-Cocont} \rightarrow \mathcal{Q}\text{-Cat}$ has a left adjoint which sends a \mathcal{Q} -category X to $J(X)$ and a \mathcal{Q} -functor f to \underline{f} .*

4. J -CONTINUOUS \mathcal{Q} -CATEGORIES

Definition 4.1. A \mathcal{Q} -category X is called *J -continuous* if the supremum $\mathcal{S}: J_S(X) \rightarrow X$ has a left adjoint.

Note that any \mathcal{Q} -functor of type $X \rightarrow J_S(X)$ corresponds to a certain \mathcal{Q} -distributor $X \dashv \rightarrow X$ belonging to J . Hence, X is J -continuous if and only if there exists a \mathcal{Q} -distributor $\Downarrow: X \dashv \rightarrow X$

necessarily in J so that, moreover, $\lceil \Downarrow \rceil$ is of type $X \rightarrow J_S(X)$ and is left adjoint to $\mathcal{S} : J_S(X) \rightarrow X$. For any \mathcal{Q} -distributor $v : X \dashrightarrow X$ one has

$$\begin{aligned} \forall \psi \in J_S(X). \lceil v \rceil \cdot \mathcal{S}(\psi) \leq \psi &\iff \forall \psi \in J_S(X), x \in X. v(x, \mathcal{S}\psi) \leq \psi(x) \\ &\iff \forall \psi \in J_S(X), x \in X. \mathcal{S}^* \cdot v(x, \psi) \leq y_*(x, \psi) \\ &\iff \mathcal{S}^* \cdot v \leq y_*. \end{aligned}$$

In particular, $\mathcal{S}^* \cdot \Downarrow \leq y_*$, and $\Downarrow : X \dashrightarrow X$ is indeed the largest such \mathcal{Q} -distributor since, for every \mathcal{Q} -distributor $v : X \dashrightarrow X$,

$$\mathcal{S}^* \cdot v \leq y_* \Rightarrow \forall x \in X. \lceil v \rceil \cdot \mathcal{S}(\lceil \Downarrow \rceil(x)) \leq \lceil \Downarrow \rceil(x) \Rightarrow \forall x \in X. \lceil v \rceil(x) \leq \lceil \Downarrow \rceil(x) \Rightarrow v \leq \Downarrow.$$

We have identified $\Downarrow : X \dashrightarrow X$ as the lifting $\Downarrow = \mathcal{S}^* \bullet y_*$ of $y_* : X \dashrightarrow J_S(X)$ along $\mathcal{S}^* : X \dashrightarrow J_S(X)$. Of course, this lifting exists in any \mathcal{Q} -category and can be studied in its own right. In 4.3 will do so, and give conditions which guarantee that it provides a left adjoint to $\mathcal{S} : J_S(X) \rightarrow X$.

Turning to the classical case $\mathcal{Q} = \mathbf{2}$ and $J = \mathbf{Idl}$, the distributor \Downarrow is given by the way-below relation. As it is well-known, the way-below relation on a continuous dcpo is the smallest approximating auxiliary relation. In what follows, we aim for a similar characterisation of the way-below \mathcal{Q} -distributor \Downarrow in the general case.

Definition 4.2. Define a \mathcal{Q} -distributor $v : X \dashrightarrow X$ to be:

- (aux) *auxiliary*, if $v \leq X$.
- (int) *interpolative*, if $v \leq v \cdot v$;
- (app) *approximating* if $v \in J$ and $X \bullet v = X$.

Furthermore, a \mathcal{Q} -distributor $v : X \dashrightarrow Y$ is:

- (cts) *Scott-continuous* if $\mathcal{S}^* \cdot v = y_* \cdot v$.

4.1. Approximating \mathcal{Q} -distributors.

Lemma 4.3. *Every approximating \mathcal{Q} -distributor $v : X \dashrightarrow X$ is auxiliary. If $v_1, v_2 : X \dashrightarrow X$ are approximating, then so is $v_2 \cdot v_1$.*

Proof. If v is approximating, then $v = X \cdot v = (X \bullet v) \cdot v \leq X$. Let now $v_1, v_2 : X \dashrightarrow X$ be approximating \mathcal{Q} -distributors. By hypothesis on J , $v_2 \cdot v_1 \in J$. Furthermore, one has

$$X \bullet (v_2 \cdot v_1) = (X \bullet v_1) \bullet v_2 = X. \quad \square$$

Proposition 4.4. *A \mathcal{Q} -distributor $v : X \dashrightarrow X$ is approximating if and only if its mate $\lceil v \rceil$ is of type $X \rightarrow J_S(X)$ and $\mathcal{S} \cdot \lceil v \rceil \cong 1_X$.*

Proof. By definition, $v : X \dashrightarrow X$ is approximating if and only if $\lceil v \rceil$ is of type $X \rightarrow J(X)$ and, for each $x \in X$, $x_* = X \bullet (x^* \cdot v)$. This in turn is equivalent to $\lceil v \rceil(x) \in J_S(X)$ and $\mathcal{S} \cdot \lceil v \rceil(x) = x$, for each $x \in X$. \square

Lemma 4.5. *Any approximating Scott-continuous \mathcal{Q} -distributor is interpolative.*

Proof. With the help of Corollary 2.9, from $\mathcal{S}^* \cdot v = y_* \cdot v$ we deduce

$$v = \lceil v \rceil^* \cdot \mathcal{S}^* \cdot v = \lceil v \rceil^* \cdot y_* \cdot v = v \cdot v. \quad \square$$

4.2. Scott-continuous \mathcal{Q} -distributors.

Proposition 4.6. *A \mathcal{Q} -distributor $v : X \dashrightarrow Y$ is Scott-continuous if and only if the \mathcal{Q} -functor $\lceil v \rceil : Y \rightarrow \widehat{X}$ is Scott-continuous.*

Proof. The \mathcal{Q} -functor $\lceil v \rceil : Y \rightarrow \widehat{X}$ is Scott-continuous if and only if, for every $\psi \in J_S(X)$, $\lceil v \rceil(\mathcal{S}(\psi)) = \text{colim}(\lceil v \rceil, \psi)$. Now just observe that

$$\text{colim}(\lceil v \rceil, \psi) = \psi \cdot v \quad (\text{by Lemma 2.11})$$

and, for every $x \in X$,

$$\begin{aligned} \psi \cdot v(x) &= \psi^* \cdot y_* \cdot v(x) = y_* \cdot v(x, \psi), & (\text{by Corollary 2.9}) \\ \lceil v \rceil(\mathcal{S}(\psi))(x) &= v(x, \mathcal{S}(\psi)) = \mathcal{S}^* \cdot v(x, \psi). & \square \end{aligned}$$

Corollary 4.7. *If $v: Y \dashrightarrow Z$ is Scott-continuous, then for any $w: X \dashrightarrow Y$, the composition $v \cdot w: X \dashrightarrow Z$ is Scott-continuous.* \square

Corollary 4.8. *A \mathcal{Q} -distributor $v: X \dashrightarrow Y$ is Scott-continuous if and only if $v \cdot x_*: X \dashrightarrow Y$ is Scott-continuous for all $x \in X$.*

Proof. From $\mathcal{S}^* \cdot v \cdot x_* = y_* \cdot v \cdot x_*$ for all $x \in X$ we deduce $\mathcal{S}^* \cdot v = y_* \cdot v$. \square

Corollary 4.9. *A \mathcal{Q} -distributor $U: 1 \dashrightarrow X$ is Scott-continuous if and only if the underlying \mathcal{Q} -functor $U: X \rightarrow \mathcal{Q}$ is Scott-continuous.*

Define

$$\Omega(X) = \{U: 1 \dashrightarrow X \mid U \text{ is Scott-continuous}\},$$

which shall be seen as a generalisation of the Scott-topology on a poset X . In fact, for $\mathcal{Q} = \mathbf{2}$ and $J = \mathbf{Idl}$, $\Omega(X)$ is the collection of all Scott-continuous maps $X \rightarrow \mathbf{2}$, which are in one-to-one correspondence with Scott-open subsets of X .

Lemma 4.10. *$\Omega(X)$ is closed under weighted colimits in \widehat{X} .*

Proof. It is enough to show that it is closed under suprema and tensors:

$$\begin{aligned} \mathcal{S}^* \cdot \bigvee_i U_i &= \bigvee_i (\mathcal{S}^* \cdot U_i) = \bigvee_i (y_* \cdot U_i) = y_* \cdot \bigvee_i U_i, \\ \mathcal{S}^* \cdot (a \otimes U) &= \mathcal{S}^* \cdot U \cdot a = y_* \cdot U \cdot a = y_* \cdot (a \otimes U). \end{aligned} \quad \square$$

For the rest of Section 4.2, assume that \mathcal{Q} is a Girard quantale. For a \mathcal{Q} -functor $f: X \rightarrow \mathcal{Q}$ we define its negation to be $(\neg f)(x) := \neg(fx)$, which is a \mathcal{Q} -functor of type $X^{op} \rightarrow \mathcal{Q}$. For a \mathcal{Q} -distributor $\varphi: X \dashrightarrow 1$, we define its negation $\neg\varphi: 1 \dashrightarrow X$ to be the \mathcal{Q} -distributor such that its underlying \mathcal{Q} -functor is the negation of the underlying \mathcal{Q} -functor for φ . Analogously we define negation for \mathcal{Q} -distributors of type $1 \dashrightarrow X$. We set:

$$\mathcal{C}(X) := \{\varphi: X \dashrightarrow 1 \mid \neg\varphi \text{ is Scott-continuous}\}.$$

Observe that $\varphi \in \Omega(X)$ if and only if $\neg\varphi \in \mathcal{C}(X)$.

Lemma 4.11. *For any $x \in X$, $yx \in \mathcal{C}(X)$.*

Proof. Let $\varphi \in J_S(X)$. Then $(yx)(\mathcal{S}\varphi) = X(x, \mathcal{S}\varphi) = \widehat{X}(yx, \varphi)$, which is equivalent to the required claim. \square

Definition 4.12. A \mathcal{Q} -functor $f: X \rightarrow Y$ is *affine* if $f^* \cdot \varphi \in \Omega(X)$ for all $\varphi \in \Omega(Y)$.

Lemma 4.13. *A \mathcal{Q} -functor $f: X \rightarrow Y$ is affine if and only if $\varphi \cdot f^* \in \mathcal{C}(X)$ for all $\varphi \in \mathcal{C}(Y)$.*

Proof. $\neg(\varphi \cdot f^*) = f^* \cdot (\neg\varphi)$. \square

Lemma 4.14. *For any \mathcal{Q} -category X :*

$$X(x, y) = \bigwedge_{\varphi \in \Omega(X)} Q(\varphi x, \varphi y),$$

for all $x, y \in X$.

Proof. For the nontrivial direction:

$$\bigwedge_{\varphi \in \Omega(X)} Q(\varphi, \varphi y) \leq Q((\neg y)(y)(x), (\neg y)(y)(y)) = Q(X(y, y), X(x, y)) = X(x, y).$$

\square

Corollary 4.15. *For any \mathcal{Q} -category X :*

$$x \leq_X y \text{ iff } \forall \varphi \in \Omega(X) \varphi x \leq \varphi y.$$

Lemma 4.16. *A \mathcal{Q} -functor $f: X \rightarrow Y$ is Scott-continuous if and only if it is affine.*

Proof. Suppose f is Scott-continuous. Then $\mathcal{S}^* \cdot f^* \cdot \varphi = \underline{f}^* \cdot \mathcal{S}^* \cdot \varphi = \underline{f}^* \cdot y_* \cdot \varphi = y_* \cdot f^* \cdot \varphi$ hold respectively by: assumption, the fact that φ is Scott-continuous and Lemma 2.10. Hence $f^* \cdot \varphi \in \Omega(X)$.

Conversely, suppose f is affine. Take any $\varphi \in J_S(X)$, set $x := \mathcal{S}\varphi$. We want to show that $fx \cong \mathcal{S}f(\varphi)$. Since for any $y \in Y$, $y(y) \in \mathcal{C}(X)$, by Lemma 4.13, $y(y) \cdot f_* \in \mathcal{C}(X)$. Thus $\neg(y(y) \cdot f_*)$ is Scott-continuous, which means that the \mathcal{Q} -functor $Y(f(-), y): X^{op} \rightarrow \mathcal{Q}$ satisfies $Y(fx, y) = \widehat{X}(\varphi, Y(f(-), y))$. In other words, $Y(fx, y) = \widehat{X}(\varphi, \widehat{f}y(y)) = \widehat{Y}(\underline{f}(\varphi), y(y))$, which proves that $fx \cong \mathcal{S}f(\varphi)$ (as they are both left adjoints to y). \square

Let $\mathcal{Q}\text{-Cat}_J$ be the category of all \mathcal{Q} -categories and Scott-continuous \mathcal{Q} -functors, and let $\mathbf{AffSet}(\mathcal{Q})$ be the category of affine sets and affine maps over \mathcal{Q} [Die99]. By the discussion above:

Theorem 4.17. *Let \mathcal{Q} be a Girard quantale. The functor $\mathcal{Q}\text{-Cat}_J \rightarrow \mathbf{AffSet}(\mathcal{Q})$ sending a \mathcal{Q} -category X to the affine set $(X, \Omega(X))$ and a Scott-continuous \mathcal{Q} -functor $f: X \rightarrow Y$ to the affine map $f: (X, \Omega(X)) \rightarrow (Y, \Omega(Y))$ is fully faithful.*

4.3. The way-below \mathcal{Q} -distributor. Motivated by the considerations at the beginning of this section, we define the *way-below \mathcal{Q} -distributor* $\Downarrow: X \dashv\!\!\dashv X$ to be the largest v such that $\mathcal{S}^* \cdot v \leq y_*$:

$$\begin{array}{ccc} J_S(X) & \xleftarrow{y_*} & X \\ \mathcal{S}^* \uparrow & \swarrow \Downarrow & \downarrow \\ X & & \end{array}$$

That is, $\Downarrow := \mathcal{S}^* \dashv\!\!\dashv y_*$.

The way-below \mathcal{Q} -distributor is not, in general, approximating; however, it is smaller than any approximating \mathcal{Q} -distributor.

Lemma 4.18. *If $v: X \dashv\!\!\dashv X$ is approximating, then $\Downarrow \leq v$. Hence, the way-below \mathcal{Q} -distributor is auxiliary.*

Proof. Since $\ulcorner v \urcorner^* \cdot y_* \leq v$, we have $y_* \leq \ulcorner v \urcorner^* \dashv\!\!\dashv v$. Hence $\Downarrow = \mathcal{S}^* \dashv\!\!\dashv y_* \leq \mathcal{S}^* \dashv\!\!\dashv (\ulcorner v \urcorner^* \dashv\!\!\dashv v) = \ulcorner v \urcorner^* \cdot \mathcal{S}^* \dashv\!\!\dashv v = X \dashv\!\!\dashv v = v$. \square

Corollary 4.19. *If \Downarrow is approximating, then \Downarrow is interpolative.*

Proof. If \Downarrow is approximating, then so is $\Downarrow \cdot \Downarrow$ and therefore $\Downarrow \leq \Downarrow \cdot \Downarrow$.

Lemma 4.20. *Any auxiliary Scott-continuous $v: X \dashv\!\!\dashv X$ satisfies $v \leq \Downarrow$.*

Proof. $\mathcal{S}^* \cdot v \leq y_* \cdot v \leq y_* \cdot X = y_*$. Therefore $v \leq \mathcal{S}^* \dashv\!\!\dashv y_* = \Downarrow$. \square

Lemma 4.21. *If $v: X \dashv\!\!\dashv X$ is interpolative and $v \leq \Downarrow$, then v is Scott-continuous.*

Proof. $v \leq \mathcal{S}^* \dashv\!\!\dashv y_*$ if and only if $\mathcal{S}^* \cdot v \leq y_*$, which yields $\mathcal{S}^* \cdot v \leq \mathcal{S}^* \cdot v \cdot v \leq y_* \cdot v$. \square

Lemma 4.22. *Let $\alpha: X \rightarrow J_S(X)$ be a Scott-continuous \mathcal{Q} -functor with $\mathcal{S}\alpha \cong 1$. Then $\alpha \dashv\!\!\dashv \mathcal{S}$.*

Proof. Let $\varphi \in J_S(X)$. Firstly, observe that $\text{colim}(\varphi, y)$ exists in $J(X)$ because it is equal to φ . However $\varphi \in J_S(X)$ and so $\text{colim}(\varphi, y) \in J_S(X)$. Secondly, since $\alpha \leq y$, we have $\text{colim}(\varphi, \alpha) \leq \text{colim}(\varphi, y)$. Taking these together we obtain $(\alpha\mathcal{S})(\varphi) \cong \text{colim}(\varphi, \alpha) \leq \text{colim}(\varphi, y) = \varphi$. \square

Theorem 4.23. *Let $v: X \dashv\!\!\dashv X \in J$. Then the following are equivalent:*

- (i) $\ulcorner v \urcorner$ is of type $X \rightarrow J_S(X)$ and $\ulcorner v \urcorner \dashv\!\!\dashv \mathcal{S}$,
- (ii) v is approximating and $v = \Downarrow$,
- (iii) v is approximating and Scott-continuous,
- (iv) v is approximating and $\ulcorner v \urcorner: X \rightarrow J_S(X)$ is Scott-continuous,
- (v) for all $x \in X$ and $\varphi \in J_S(X)$ we have $\widehat{X}(\ulcorner v \urcorner(x), \varphi) = X(x, \mathcal{S}\varphi)$.

Proof. The implication (i) \Rightarrow (ii) we have already discussed at the beginning of this section. To see (ii) \Rightarrow (iii), assume that \Downarrow is approximating. Then \Downarrow is interpolative and therefore Scott-continuous. Assume now (iii). Then $\lceil v \rceil: X \rightarrow \widehat{X}$ is Scott-continuous. Therefore also $\lceil v \rceil: X \rightarrow J_S(X)$, which shows that (iii) \Rightarrow (iv). Proposition 4.4 and Lemma 4.22 imply immediately (iv) \Rightarrow (i). Clearly, (i) implies (v). Finally, assume (v). Then

$$(X \bullet - v)(x, y) = \widehat{X}(\lceil v \rceil(x), \lceil X \rceil(y)) = \widehat{X}(\lceil v \rceil(x), y(y)) = X(x, \mathcal{S}y(y)) = X(x, y),$$

which proves that v is approximating. Hence, $\lceil v \rceil$ is of type $X \rightarrow J_S(X)$ and indeed left adjoint to \mathcal{S} . \square

Theorem 4.24. *The following are equivalent, for a \mathcal{Q} -category X .*

- (i) X is J -continuous,
- (ii) The way-below \mathcal{Q} -distributor $\Downarrow: X \multimap X$ is approximating,
- (iii) There exists a Scott-continuous approximating \mathcal{Q} -distributor $v: X \multimap X$.

4.4. Continuous versus projective. Employing ideas from [Stu07], we show now that continuity of a J -cocomplete category X corresponds exactly to projectivity with respect to a certain class of epimorphisms in the category $\mathbf{J-Cocont}$ of J -cocomplete \mathcal{Q} -categories and Scott-continuous \mathcal{Q} -functors. More precisely, we call X in $\mathbf{J-Cocont}$ *J -projective* if, for every Scott-continuous $h: A \rightarrow B$ which has a right inverse in $\mathcal{Q}\text{-Cat}$ (i.e., there exists a \mathcal{Q} -functor $e: B \rightarrow A$ with $h \cdot e \cong 1_B$) and for any morphism $f: X \rightarrow B$ there exists a morphism $g: X \rightarrow A$ that lifts f , i.e. $h \cdot g \cong f$:

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \uparrow g & \nearrow f & \\ X & & \end{array}$$

Lemma 4.25. *$J(X)$ is projective.*

Proof. Let $h: A \rightarrow B$ be a surjective Scott-continuous \mathcal{Q} -functor which admits a right inverse \mathcal{Q} -functor $e: B \rightarrow A$, and let $f: J(X) \rightarrow B$ be Scott-continuous. The unique Scott-continuous extension $j: J(X) \rightarrow B$ of $efy: X \rightarrow B$ (compare Prop. 3.8) is the required mapping. \square

Clearly, retracts of projective objects are projective. Hence we have:

Theorem 4.26. *A J -cocomplete category X is J -continuous if and only if it is J -projective in $\mathbf{J-Cocont}$.*

Proof. For the remaining direction, suppose X is projective. Consider the commutative diagram:

$$\begin{array}{ccc} J(X) & \xrightarrow{\mathcal{S}} & X \\ \uparrow \alpha & \nearrow 1_X & \\ X & & \end{array}$$

The vertical map α is chosen to be the lifting of the identity 1_X . Since $\mathcal{S}\alpha \cong 1_X$, Lemma 4.22 and Theorem 4.23 yield that X is J -continuous. \square

5. J -CONTINUOUS $(\mathbb{U}, \mathcal{Q})$ -CATEGORIES

We propose a further substantial generalisation of continuous partial orders that is introduced by adding to the choice of a quantale \mathcal{Q} , a class of distributors J , also a choice of a monad \mathbb{T} on \mathbf{Set} . Formally speaking, we seek to define J -continuous $(\mathbb{T}, \mathcal{Q})$ -categories, thus achieving a greater applicability of our theory. We recall that the category of $(\mathbb{T}, \mathcal{Q})$ -categories is: (i) isomorphic to $\mathcal{Q}\text{-Cat}$ for \mathbb{T} being the identity monad; (ii) isomorphic to \mathbf{Top} , for $\mathcal{Q} = \mathbf{2}$ and $\mathbb{T} = \mathbb{U}$ – the ultrafilter monad.

In this paper we decided to limit our choice of monad to \mathbb{U} (the identity monad case already implicitly discussed in preceding sections) but we hasten to remark that the majority of the results that follow can be restated and proved in the general setting.

5.1. **The ultrafilter monad.** The ultrafilter monad $\mathbb{U} = (U, e, m)$ consists of:

- a functor $U: \mathbf{Set} \rightarrow \mathbf{Set}$ that to each set X assigns the set of all ultrafilters on X , and to each map $f: X \rightarrow Y$ assigns a map $Uf: UX \rightarrow UY$ given by $Uf(\sigma) := \{B \subseteq Y \mid f^{-1}[B] \in \sigma\}$;
- the identity e , which is a natural transformation from the identity functor on \mathbf{Set} to U given componentwise by: $e_X: X \rightarrow UX$, $e_X(x) := \dot{x} = \{A \subseteq X \mid x \in A\}$;
- the multiplication m , which is a natural transformation of type $UU \rightarrow U$. Its component $m_X: UUX \rightarrow UX$ assigns to each ultrafilter of ultrafilters Υ these subsets A of X for which $A^\# = \{\sigma \in UX \mid A \in \sigma\}$ belongs to Υ .

5.2. **The Lawson topology on \mathcal{Q} .** In order to make the ultrafilter monad structure compatible with the quantale structure we consider a compact Hausdorff topology on \mathcal{Q} . For a large class of quantales, e.g. for all quantales, which have infimum as tensor, or for $[0, 1]$ and $[0, \infty]$, the infimum $\wedge: \mathcal{Q} \times \mathcal{Q} \rightarrow \mathcal{Q}$ is Lawson-continuous. Therefore by the Fundamental Theorem of Compact Semilattices (Theorem VI-3.4 of [GHK+03]), the only compact Hausdorff topology on \mathcal{Q} making \wedge continuous in the Lawson topology. Hence, to keep the presentation simple, it seems reasonable to restrict our attention only to quantales with Lawson-continuous infimum operation and *from now on we assume so*.

In our paper the Lawson topology will be referred to via its induced convergence. Note that by Thm.III-3.17 of [GHK+03] its convergence is a map $\xi: U\mathcal{Q} \rightarrow \mathcal{Q}$ defined as $\xi(\sigma) := \bigvee_{A \in \sigma} \bigwedge A$. Since $\lambda(\mathcal{Q})$ is compact Hausdorff, every ultrafilter converges to exactly one point on \mathcal{Q} , i.e. the following diagrams commute:

$$\begin{array}{ccc} Q & \xrightarrow{e_Q} & UQ \\ & \searrow 1_Q & \downarrow \xi \\ & & Q \end{array} \qquad \begin{array}{ccc} UUQ & \xrightarrow{U\xi} & UQ \\ m_Q \downarrow & & \downarrow \xi \\ UQ & \xrightarrow{\xi} & Q \end{array}$$

This means exactly that ξ is an \mathbb{U} -algebra structure on \mathcal{Q} . Moreover, $\mathbf{1}: \mathbf{1} \rightarrow \mathcal{Q}$ is clearly Lawson-continuous and we assume $\otimes: \mathcal{Q} \times \mathcal{Q} \rightarrow \mathcal{Q}$ to be Lawson-continuous as well. In this case both maps are \mathbb{U} -algebra homomorphisms, that is:

$$\begin{array}{ccc} U\mathbf{1} & \xrightarrow{U\mathbf{1}} & UQ \\ \downarrow \mathbf{!} & & \downarrow \xi \\ \mathbf{1} & \xrightarrow{\mathbf{1}} & Q \end{array} \qquad \begin{array}{ccc} U(Q \times Q) & \xrightarrow{U(\otimes)} & UQ \\ \langle \xi \cdot U\pi_1, \xi \cdot U\pi_2 \rangle \downarrow & & \downarrow \xi \\ Q \times Q & \xrightarrow{\otimes} & Q \end{array}$$

commute. We also require the following technical property: whenever for $f: X \rightarrow Y$, $\varphi: X \rightarrow \mathcal{Q}$, $\psi: Y \rightarrow \mathcal{Q}$ we have $\psi(y) \leq \bigvee_{\{x \mid f x = y\}} \varphi(x)$, then $\xi(U(\psi)(\sigma)) \leq \bigvee_{\{\nu \mid U(f)(\nu) = \sigma\}} \xi(U(\varphi)(\nu))$ holds. Finally, if \mathcal{Q} is a Girard quantale, then for any ultrafilter ν we have $\xi(\nu) = \neg \bigvee_{A \in \nu} \bigwedge_{x \in A} x = \bigwedge_{A \in \nu} \bigvee_{x \in A} \neg x = \bigvee_{A \in \nu} \bigwedge_{x \in A} \neg x = \xi(U(\neg)(\nu))$. This means that the negation considered as a map $\neg: \mathcal{Q} \rightarrow \mathcal{Q}$ is Lawson-continuous and hence is the \mathbb{U} -algebra homomorphism:

$$\begin{array}{ccc} UQ & \xrightarrow{U(\neg)} & UQ \\ \xi \downarrow & & \downarrow \xi \\ Q & \xrightarrow{\neg} & Q \end{array}$$

In conclusion, the triple $(\mathbb{U}, \mathcal{Q}, \xi)$ is a strict topological theory in the sense of [Hof07]. In the following subsections we summarise the main aspects of the theory of $(\mathbb{U}, \mathcal{Q})$ -categories, referring to [CT03, Hof07, Hof08, CH08] for further details. We remark that many notions and results do not differ dramatically from the \mathcal{Q} -case, with the notable exception of the dual category and, consequently, the Yoneda lemma (see Proposition 5.4, Lemma 5.5 and Corollary 5.6 below). Our main contribution here is the introduction and study of continuity (see Subsection 5.7), which has to face yet another problem: the lifting of distributors is not always available in the

(\mathbb{U}, \mathcal{Q})-case. Therefore we cannot use freely the way-below distributor \Downarrow , however, we prove that local versions of \Downarrow do exist and can often be used instead.

5.3. (\mathbb{U}, \mathcal{Q})-relations. The functor $U: \mathbf{Set} \rightarrow \mathbf{Set}$ extends to $U^+: \mathcal{Q}\text{-Rel} \rightarrow \mathcal{Q}\text{-Rel}$ as follows: we put $U^+X = UX$ for each set X , and $U^+: UX \times UY \rightarrow \mathcal{Q}$ is:

$$r(\sigma, \nu) \mapsto \bigvee \{ \xi \cdot Ur(\omega) \mid \omega \in U(X \times Y), U\pi_1(\omega) = \sigma, U\pi_2(\omega) = \nu \}$$

for each \mathcal{Q} -relation $r: X \multimap Y$. As shown in [Hof07], we have $U^+f = Uf$ for each function $f: X \rightarrow Y$, $U^+(r^\circ) = U^+(r)^\circ$ (and so we write U^+r°) for each \mathcal{Q} -relation $r: X \multimap Y$.

A \mathcal{Q} -relation of the form $\alpha: UX \multimap Y$ we call (\mathbb{U}, \mathcal{Q})-relation from X to Y , and write $\alpha: X \multimap Y$. For (\mathbb{U}, \mathcal{Q})-relations $\alpha: X \multimap Y$ and $\beta: Y \multimap Z$ we define the Kleisli convolution $\beta \circ \alpha: X \multimap Z$ as $\beta \circ \alpha = \beta \cdot U^+\alpha \cdot m_X^\circ$. Kleisli convolution is associative and has the (\mathbb{U}, \mathcal{Q})-relation $e_X^\circ: X \multimap X$ as a lax identity: $a \circ e_X^\circ = a$ and $e_Y^\circ \circ a \geq a$ for any $a: X \multimap Y$. We call $a: X \multimap Y$ unitary if $e_Y^\circ \circ a = a$. Furthermore, for a (\mathbb{U}, \mathcal{Q})-relation $\alpha: X \multimap Y$, the composition function $(-) \circ \alpha$ still has a right adjoint $(-) \circ \alpha$ (we define $\gamma \circ \alpha := \gamma \bullet (U^+(\alpha) \cdot m_X^\circ)$) but $\alpha \circ (-)$ in general does not.

5.4. (\mathbb{U}, \mathcal{Q})-categories. An (\mathbb{U}, \mathcal{Q})-category is a pair consisting of a set X and an (\mathbb{U}, \mathcal{Q})-endorelation $X(-, -): X \multimap X$ such that $e_X^\circ \leq X$ and $X \circ X \leq X$. Expressed elementwise, these conditions become: $\mathbf{1} \leq X(e_X(x), x)$ and $U^+X(\Upsilon, v) \otimes X(v, x) \leq X(m_X(\Upsilon), x)$ for all $\Upsilon \in UUX$, $v \in UX$ and $x \in X$. A function $f: X \rightarrow Y$ between (\mathbb{U}, \mathcal{Q})-categories is a (\mathbb{U}, \mathcal{Q})-functor if $f \cdot X \leq Y \cdot Tf$, which in pointwise notation reads as $X(v, x) \leq Y(Uf(v), f(x))$ for all $v \in UX$, $x \in X$. If we have above even equality, we call $f: X \rightarrow Y$ fully faithful. The resulting category of (\mathbb{U}, \mathcal{Q})-categories and (\mathbb{U}, \mathcal{Q})-functors we denote as (\mathbb{U}, \mathcal{Q})-**Cat**. The quantale \mathcal{Q} becomes a (\mathbb{U}, \mathcal{Q})-category $\mathcal{Q} = (\mathcal{Q}, \text{hom}_\xi)$, where $\text{hom}_\xi: U\mathcal{Q} \times \mathcal{Q} \rightarrow \mathcal{Q}$, $(\sigma, v) \mapsto \text{hom}(\xi(\sigma), v)$. By $|X|$ we denote the (\mathbb{U}, \mathcal{Q})-category (UX, m_X) . There is also a free (\mathbb{U}, \mathcal{Q})-category on a set X given by (X, e_X°) . We have a canonical forgetful functor $S: (\mathbb{U}, \mathcal{Q})\text{-Cat} \rightarrow \mathcal{Q}\text{-Cat}$ sending a (\mathbb{U}, \mathcal{Q})-category X to its underlying \mathcal{Q} -category $SX = (X, X \cdot e_X)$. Furthermore, S has a left adjoint $A: \mathcal{Q}\text{-Cat} \rightarrow (\mathbb{U}, \mathcal{Q})\text{-Cat}$ defined by $AX = (X, e_X^\circ \cdot U^+X)$, for each \mathcal{Q} -category X .

Example 5.1. For $\mathcal{Q} = \mathbf{2}$, a (\mathbb{U}, \mathcal{Q})-category is a topological space presented via its ultrafilter convergence structure, and a function $f: X \rightarrow Y$ between topological spaces is continuous if and only if it is (\mathbb{U}, \mathcal{Q})-functor (see [Bar70]). The functor $S: \mathbf{Top} \rightarrow \mathbf{Ord}$ sends a topological space X to the ordered set X where $x \leq y : \iff \dot{x} \rightarrow y \iff y \in \overline{\{x\}}$. Note that we consider X here with the dual of the specialization order.

Example 5.2. For $\mathcal{Q} = [0, \infty]$, (\mathbb{U}, \mathcal{Q})-categories correspond to approach spaces and (\mathbb{U}, \mathcal{Q})-functors to non-expansive maps (see [Low97]).

There is yet another functor connecting (\mathbb{U}, \mathcal{Q})-categories with \mathcal{Q} -categories, namely $M: (\mathbb{U}, \mathcal{Q})\text{-Cat} \rightarrow \mathcal{Q}\text{-Cat}$ which sends a (\mathbb{U}, \mathcal{Q})-category X to the \mathcal{Q} -category $(UX, U^+X \cdot m_X^\circ)$. These functors are all needed to define the dual of a (\mathbb{U}, \mathcal{Q})-category X , namely $X^{\text{op}} := A((MX)^{\text{op}})$.

As studied in [Hof07] the tensor product of \mathcal{Q} can be transported to (\mathbb{U}, \mathcal{Q})-**Cat** by putting $X \otimes Y := X \times Y$ with structure $(X \otimes Y)(\sigma, (x, y)) = X(v, x) \otimes Y(\nu, y)$, where $\sigma \in U(X \times Y)$, $x \in X$, $y \in Y$, $v = U\pi_1(\sigma)$ and $\nu = U\pi_2(\sigma)$. The (\mathbb{U}, \mathcal{Q})-category $E = (\mathbf{1}, \mathbf{1})$ is a \otimes -neutral object, where $\mathbf{1}$ is a singleton set and $\mathbf{1}: U\mathbf{1} \times \mathbf{1} \rightarrow \mathcal{Q}$ the constant relation with value $\mathbf{1} \in \mathcal{Q}$. In general, this constructions does not result in a closed structure on (\mathbb{U}, \mathcal{Q})-**Cat**; however we have that: $|X| \otimes (-): (\mathbb{U}, \mathcal{Q})\text{-Cat} \rightarrow (\mathbb{U}, \mathcal{Q})\text{-Cat}$ has a right adjoint $(-)^{|X|}: (\mathbb{U}, \mathcal{Q})\text{-Cat} \rightarrow (\mathbb{U}, \mathcal{Q})\text{-Cat}$. The structure on $\mathcal{Q}^{|X|}$ is given by the formula

$$\mathcal{Q}^{|X|}(\rho, \psi) = \bigwedge_{\{\varrho \in T(|X| \times \mathcal{Q}^{|X|}) \mid \varrho \rightarrow \rho\}} \mathcal{Q}(\xi \cdot U \text{ev}(\varrho), \psi(m_X \cdot U\pi_1(\varrho))),$$

for each $\rho \in U\mathcal{Q}^{|X|}$ and $\psi \in \mathcal{Q}^{|X|}$. In particular for $\rho = e_{\mathcal{Q}^{|X|}}(\varphi)$ we have:

$$\mathcal{Q}^{|X|}(e_{\mathcal{Q}^{|X|}}(\varphi), \psi) = \bigwedge_{v \in UX} \mathcal{Q}(\varphi v, \psi v).$$

5.5. **(\mathbb{U}, \mathcal{Q})-distributors.** Let X and Y be $(\mathbb{U}, \mathcal{Q})$ -categories and $\varphi: X \multimap Y$ be a $(\mathbb{U}, \mathcal{Q})$ -relation. We call φ a $(\mathbb{U}, \mathcal{Q})$ -distributor, and write $\varphi: X \multimap Y$, if $\varphi \circ X = \varphi$ and $Y \circ \varphi = \varphi$. Kleisli convolution is associative, and it follows that $\psi \circ \varphi$ is a $(\mathbb{U}, \mathcal{Q})$ -distributor if $\psi: Y \multimap Z$ and $\varphi: X \multimap Y$ are so. Furthermore, we have $X(-, -): X \multimap X$ for each $(\mathbb{U}, \mathcal{Q})$ -category X , and, by definition, $X(-, -)$ is the identity $(\mathbb{U}, \mathcal{Q})$ -distributor on X for the Kleisli convolution. In other words, $(\mathbb{U}, \mathcal{Q})$ -categories and $(\mathbb{U}, \mathcal{Q})$ -distributors form a category, denoted as $(\mathbb{U}, \mathcal{Q})$ -**Dist**, with Kleisli convolution as compositional structure. In fact, $(\mathbb{U}, \mathcal{Q})$ -**Dist** is an ordered category with the structure on hom-sets inherited from $(\mathbb{U}, \mathcal{Q})$ -**Rel**. Finally, a $(\mathbb{U}, \mathcal{Q})$ -relation $\varphi: X \multimap Y$ is unitary precisely if φ is a $(\mathbb{U}, \mathcal{Q})$ -distributor $\varphi: (X, e_X^\circ) \multimap (Y, e_Y^\circ)$ between the corresponding discrete $(\mathbb{U}, \mathcal{Q})$ -categories.

Let now $f: X \rightarrow Y$ be a function between $(\mathbb{U}, \mathcal{Q})$ -categories. We define $(\mathbb{U}, \mathcal{Q})$ -relations $f_*: X \multimap Y$ and $f^*: Y \multimap X$ by putting $f_* = Y \cdot Uf$ and $f^* = f^\circ \cdot Y$ respectively. Hence, for $\sigma \in UX$, $\nu \in UY$, $x \in X$ and $y \in Y$, we have $f_*(\sigma, y) = b(Uf(\sigma), y)$ and $f^*(\nu, x) = b(\nu, f(x))$. Given now $(\mathbb{U}, \mathcal{Q})$ -distributors φ and ψ , we easily obtain $\varphi \circ f_* = \varphi \cdot Uf$ and $f^* \circ \psi = f^\circ \cdot \psi$. In particular, $Y \circ f_* = f_*$ and $f^* \circ Y = f^*$. One can then show that

Lemma 5.3. *The following assertions are equivalent:*

- (i) $f: X \rightarrow Y$ is a $(\mathbb{U}, \mathcal{Q})$ -functor;
- (ii) f_* is a $(\mathbb{U}, \mathcal{Q})$ -distributor $f_*: X \multimap Y$;
- (iii) f^* is a $(\mathbb{U}, \mathcal{Q})$ -distributor $f^*: Y \multimap X$.

For each $(\mathbb{U}, \mathcal{Q})$ -functor $f: X \rightarrow Y$ we have, in fact, an adjunction $f_* \dashv f^*$ in $(\mathbb{U}, \mathcal{Q})$ -**Dist**. Moreover, given a $(\mathbb{U}, \mathcal{Q})$ -functor $g: Y \rightarrow Z$, $g_* \circ f_* = (g \cdot f)_*$ and $f^* \circ g^* = (g \cdot f)^*$, plus $(1_X)_* = (1_X)^* = X$.

We will often need the following crucial property.

Proposition 5.4. *For an $(\mathbb{U}, \mathcal{Q})$ -relation $\psi: X \multimap Y$, the following are equivalent:*

- (i) $\psi: X \multimap Y$ is a $(\mathbb{U}, \mathcal{Q})$ -distributor;
- (ii) Both $\psi: |X| \otimes Y \rightarrow \mathcal{Q}$ and $\psi: X^{\text{op}} \otimes Y \rightarrow \mathcal{Q}$ are $(\mathbb{U}, \mathcal{Q})$ -functors.

Therefore, each $(\mathbb{U}, \mathcal{Q})$ -distributor $\varphi: X \multimap Y$ defines a $(\mathbb{U}, \mathcal{Q})$ -functor $\ulcorner \varphi \urcorner: Y \rightarrow \mathcal{Q}^{|X|}$ which factors through the embedding $\widehat{X} \hookrightarrow \mathcal{Q}^{|X|}$, where $\widehat{X} = \{\psi \in \mathcal{Q}^{|X|} \mid \psi: X \multimap 1\}$ and 1 denotes the $(\mathbb{U}, \mathcal{Q})$ -category $(1, e_1^\circ)$.

$$\begin{array}{ccc} Y & \xrightarrow{\ulcorner \varphi \urcorner} & \mathcal{Q}^{|X|} \\ & \searrow \ulcorner \varphi \urcorner & \uparrow \\ & & \widehat{X} \end{array}$$

In particular, for each $(\mathbb{U}, \mathcal{Q})$ -category X we have $X(-, -): X \multimap X$, and therefore obtain the Yoneda $(\mathbb{U}, \mathcal{Q})$ -functor $y = \ulcorner X \urcorner: X \rightarrow \widehat{X}$. The following result is crucial to transport \mathcal{Q} -categorical ideas into the $(\mathbb{U}, \mathcal{Q})$ -setting.

Lemma 5.5. *Let $\psi: X \multimap Z$ and $\varphi: X \multimap Y$ be $(\mathbb{U}, \mathcal{Q})$ -distributors. Then, for all $\zeta \in UZ$ and $y \in Y$, $\mathcal{Q}^{|X|}(U\ulcorner \psi \urcorner(\zeta), \ulcorner \varphi \urcorner(y)) = (\varphi \circ \psi)(\zeta, y)$.*

Corollary 5.6. (1) *For each $\varphi \in \widehat{X}$ and each $\sigma \in UX$, $\varphi(\sigma) = \mathcal{Q}^{|X|}(Uy(\sigma), \varphi)$, that is, $(y)_*: X \multimap \widehat{X}$ is given by the evaluation map $\text{ev}: UX \times \widehat{X} \rightarrow \mathcal{Q}$. As a consequence, $y: X \rightarrow \widehat{X}$ is fully faithful.*

(2) *For every $(\mathbb{U}, \mathcal{Q})$ -distributor $\varphi: X \multimap Y$, $\ulcorner \varphi \urcorner^* \circ y_* = \varphi$.*

5.6. **J -cocomplete $(\mathbb{U}, \mathcal{Q})$ -categories.** The notion of colimit can be easily transported into the realm of $(\mathbb{U}, \mathcal{Q})$ -categories: just substitute the relational composition of \mathcal{Q} -distributors by Kleisli-composition of $(\mathbb{U}, \mathcal{Q})$ -distributors [Hof08, CH08]. More precisely, a weighted diagram in a $(\mathbb{U}, \mathcal{Q})$ -category X is given by a $(\mathbb{U}, \mathcal{Q})$ -functor $d: D \rightarrow X$ and a weight $\varphi: D \multimap A$. A *colimit* of such a diagram is a $(\mathbb{U}, \mathcal{Q})$ -functor $g: A \rightarrow X$ which represents $d_* \circ \varphi$, that is, $g_* = d_* \circ \varphi$. One calls g a φ -weighted colimit of d , and writes $g \simeq \text{colim}(d, \varphi)$. A $(\mathbb{U}, \mathcal{Q})$ -functor $f: X \rightarrow Y$ preserves the φ -weighted colimit g of d if $f \cdot g$ is the φ -weighted colimit of $f \cdot d$, that is, if $(f \cdot g)_* = (f \cdot d)_* \circ \varphi$. The following is well-known for $(\mathbb{U}, \mathcal{Q})$ -categories:

Lemma 5.7. For $(\mathbb{U}, \mathcal{Q})$ -distributors $\varphi: X \dashrightarrow Y$ and $\psi: Y \dashrightarrow Z$, $\lceil \psi \cdot \varphi \rceil \cong \text{colim}(\lceil \varphi \rceil, \psi)$. In particular, every $\varphi \in \widehat{X}$ is the φ -weighted colimit of $y: X \rightarrow \widehat{X}$.

As in the case of \mathcal{Q} -categories, we consider cocompleteness and continuity with respect to chosen distributors. To do so, let J be a subcategory of $(\mathbb{U}, \mathcal{Q})\text{-Dist}$ such that, for every $(\mathbb{U}, \mathcal{Q})$ -functor $f, f^* \in J$ and, for all $\varphi: X \dashrightarrow Y \in (\mathbb{U}, \mathcal{Q})\text{-Dist}$,

$$(\forall y \in Y. y^* \circ \varphi \in J) \Rightarrow \varphi \in J.$$

A $(\mathbb{U}, \mathcal{Q})$ -category X is J -cocomplete if all J -weighted colimits exist in X . Furthermore, one calls a $(\mathbb{U}, \mathcal{Q})$ -functor f J -dense if $f_* \in J$. A $(\mathbb{U}, \mathcal{Q})$ -category X is called J -injective if, for every J -dense embedding $i: B \rightarrow Y$ and each $(\mathbb{U}, \mathcal{Q})$ -functor $f: B \rightarrow X$, there is a $(\mathbb{U}, \mathcal{Q})$ -functor $g: Y \rightarrow X$ with $g \cdot i \cong f$. From [CH08] we import

Theorem 5.8. The following assertions are equivalent, for a $(\mathbb{U}, \mathcal{Q})$ -category X .

- (i) X is J -injective.
- (ii) $y: X \rightarrow J(X)$ has a left inverse $\mathcal{S}: J(X) \rightarrow X$.
- (iii) $y: X \rightarrow J(X)$ has a left adjoint $\mathcal{S}: J(X) \rightarrow X$.
- (iv) X is J -cocomplete.

5.7. J -continuous $(\mathbb{U}, \mathcal{Q})$ -categories. We come now to our main purpose in this section and introduce J -continuous $(\mathbb{U}, \mathcal{Q})$ -categories. Naturally, we define

Definition 5.9. A $(\mathbb{U}, \mathcal{Q})$ -category X is called J -continuous if the $(\mathbb{U}, \mathcal{Q})$ -functor $\mathcal{S}: J_{\mathcal{S}}(X) \rightarrow X$ has a left adjoint.

As in the \mathcal{Q} -case, such a left adjoint $(\mathbb{U}, \mathcal{Q})$ -functor $X \rightarrow J_{\mathcal{S}}(X)$ corresponds to a $(\mathbb{U}, \mathcal{Q})$ -distributor $\Downarrow: X \dashrightarrow X$ which necessarily belongs to J and, moreover, must be the lifting $\Downarrow = \mathcal{S}^* \dashv y_*$ of $y_*: X \dashrightarrow J_{\mathcal{S}}(X)$ along $\mathcal{S}^*: X \dashrightarrow J_{\mathcal{S}}(X)$. However, an immediate problem in generalizing the way-below relation to an $(\mathbb{U}, \mathcal{Q})$ -distributor in an analogous way to the \mathcal{Q} -distributor case stems from the fact that in general the lifting \dashv between $(\mathbb{U}, \mathcal{Q})$ -distributors does not exist. We deal first with this problem.

Lemma 5.10. Let $\psi: UY \dashrightarrow X$ and $\varphi: Z \dashrightarrow X$ be \mathcal{Q} -relations, and let $\varphi \dashv \bullet \psi: UY \dashrightarrow Z$ the lifting of ψ along φ in $\mathcal{Q}\text{-Rel}$.

$$\begin{array}{ccc} X & \xleftarrow{\psi} & UY \\ \varphi \uparrow & & \swarrow \varphi \dashv \bullet \psi \\ Z & & \end{array}$$

If ψ is a unitary $(\mathbb{U}, \mathcal{Q})$ -relation $\psi: Y \dashrightarrow X$, then so is $\varphi \dashv \bullet \psi: Y \dashrightarrow Z$.

Proof. We have to show that

$$\varphi \dashv \bullet \psi: |Y| \otimes Z_D \rightarrow \mathcal{Q}, (\nu, z) = \bigwedge_{x \in X} \mathcal{Q}(\varphi(z, x), \psi(\nu, x))$$

is a $(\mathbb{U}, \mathcal{Q})$ -functor, where Z_D denotes the free $(\mathbb{U}, \mathcal{Q})$ -category $Z_D = (Z, e_Z^o)$ on the set Z . Since $\bigwedge: \mathcal{Q}^{X_D} \rightarrow \mathcal{Q}$ is a $(\mathbb{U}, \mathcal{Q})$ -functor, it is enough to show functoriality of

$$|Y| \otimes Z_D \otimes X_D \rightarrow \mathcal{Q}, (\nu, z, x) = \mathcal{Q}(\varphi(z, x), \psi(\nu, x)).$$

But this function can be expressed as a composite of $(\mathbb{U}, \mathcal{Q})$ -functors

$$|Y| \otimes Z_D \otimes X_D \rightarrow Z_D \otimes X_D \otimes |Y| \otimes X_D \xrightarrow{\varphi \otimes \psi} \mathcal{Q}_D \otimes \mathcal{Q} \rightarrow \mathcal{Q}.$$

Note that we use here symmetry of the tensor product \otimes and functoriality of $\Delta_X: X_D \rightarrow X_D \otimes X_D$. \square

Lemma 5.11. Let $\varphi: X \dashrightarrow Y$ and $\psi: Y \dashrightarrow Z$ be $(\mathbb{U}, \mathcal{Q})$ -relations. Furthermore, assume that φ is unitary and Y finite. Then $\psi \circ \varphi = \psi \cdot e_Y \cdot \varphi$.

Proof. Just observe that

$$\psi \circ \varphi = \psi \cdot U^+ \varphi \cdot m_X^o = \psi \cdot e_Y \cdot e_Y^o \cdot U^+ \varphi \cdot m_X^o = \psi \cdot e_Y \cdot \varphi. \quad \square$$

Lemma 5.12. For all $(\mathbb{U}, \mathcal{Q})$ -distributors $\varphi: Y \multimap X$ and $\psi: 1 \multimap X$, φ has a lifting along an ψ in $(\mathbb{U}, \mathcal{Q})$ -**Dist** which is given by $\psi \multimap \varphi = \psi \cdot e_1 \multimap \varphi$.

$$\begin{array}{ccc} X & \xleftarrow{\varphi} & Y \\ \psi \downarrow & \circlearrowleft & \swarrow \psi \multimap \varphi \\ 1 & & \end{array}$$

Proof. Let $\gamma: Y \multimap 1$ be an unitary $(\mathbb{U}, \mathcal{Q})$ -relation. Then $\psi \circ \gamma \leq \varphi$ if and only if $\psi \cdot e_1 \cdot \gamma \leq \varphi$ if and only if $\gamma \leq \psi \cdot e_1 \multimap \varphi$. \square

By analogy with \mathcal{Q} -distributors, define $v: X \multimap X$ to be

- *auxiliary*, if $v \leq X$;
- *approximating*, if: $v \in J$, and $X \multimap v = X$;
- *interpolative*, if $v \leq v \circ v$.

We call a $(\mathbb{U}, \mathcal{Q})$ -distributor $v: X \multimap Y$

- *Scott-continuous* if $\mathcal{S}^* \circ v = y_* \circ v$.

Any approximating $(\mathbb{U}, \mathcal{Q})$ -distributor is auxiliary, and any approximating Scott-continuous $(\mathbb{U}, \mathcal{Q})$ -distributor is interpolative. Furthermore, the composition of approximating $(\mathbb{U}, \mathcal{Q})$ -distributors is again approximating (compare with Lemmata 4.3 and 4.5).

With the same proof as for Proposition 4.6 one verifies that $v: X \multimap Y$ is Scott-continuous if and only if the $(\mathbb{U}, \mathcal{Q})$ -functor $\lceil v \rceil: Y \rightarrow \widehat{X}$ is Scott-continuous.

We also define the way-below $(\mathbb{U}, \mathcal{Q})$ -distributor $\Downarrow: X \multimap X$ as the lifting of $y_*: X \multimap J_S(X)$ along $\mathcal{S}^*: X \multimap J_S(X)$, whenever it exists. Since we do not have in general the way-below distributor ‘globally’, we define its ‘local’ version at $x \in X$ to be the lifting of y_* along $\mathcal{S}^* \circ x_*$,

$$\begin{array}{ccc} J_S(X) & \xrightarrow{y_*} & X \\ \mathcal{S}^* \circ x_* \downarrow & \circlearrowleft & \swarrow \Downarrow_x := (\mathcal{S}^* \circ x_*) \multimap y_* \\ 1 & & \end{array}$$

which does exist for each $(\mathbb{U}, \mathcal{Q})$ -category X and each $x \in X$. Of course, if \Downarrow exists on X , then $\Downarrow_x = x^* \circ \Downarrow$ for each $x \in X$.

Lemma 5.13. For every $(\mathbb{U}, \mathcal{Q})$ -category X , the map $\Downarrow_-: X \rightarrow \widehat{X}$, $x \mapsto \Downarrow_x$ is a \mathcal{Q} -functor.

Proof. For any $x, y \in X$, we have to show that

$$X(x, y) \leq \Downarrow_y \multimap \Downarrow_x.$$

First note that $X(x, y) = y^* \circ x_*$. Now,

$$y^* \circ x_* \leq (\mathcal{S}^* \circ y_* \multimap y_*) \multimap (\mathcal{S}^* \circ x_* \multimap y_*)$$

if and only if

$$y^* \circ x_* \circ (\mathcal{S}^* \circ x_* \multimap y_*) \leq \mathcal{S}^* \circ y_* \multimap y_*,$$

which in turn is equivalent to

$$\mathcal{S}^* \circ y_* \circ y^* \circ x_* \circ (\mathcal{S}^* \circ x_* \multimap y_*) \leq y_*;$$

and this is indeed true since $y_* \circ y^* \leq X$. \square

So far we are not able to prove or disprove that \Downarrow_- is a $(\mathbb{U}, \mathcal{Q})$ -functor. Of course, \Downarrow_- is a $(\mathbb{U}, \mathcal{Q})$ -functor if X is J -continuous, since in this case $\Downarrow_- = \lceil \Downarrow \rceil$.

Proposition 5.14. A $(\mathbb{U}, \mathcal{Q})$ -category X is J -continuous if and only if \Downarrow_- is a $(\mathbb{U}, \mathcal{Q})$ -functor and, for each $x \in X$, $\Downarrow_x \in J(X)$ and $X \multimap \Downarrow_x = x_*$.

Proof. Clearly, the conditions are necessary. Assume now that \Downarrow_- is a $(\mathbb{U}, \mathcal{Q})$ -functor and $\Downarrow_x \in J(X)$ and $X \circ - \Downarrow_x = x_*$ for each $x \in X$. Hence \Downarrow_- is of type $X \rightarrow J_S(X)$ and we have $\mathcal{S} \cdot \Downarrow_x \cong x$. Let now $\psi \in J_S(X)$. Then

$$\Downarrow_{\mathcal{S}\psi}(\nu) = \bigwedge_{\varphi \in J_S(X)} Q(X(\mathcal{S}\psi, \mathcal{S}\varphi), \varphi(\nu)) \leq Q(X(\mathcal{S}\psi, \mathcal{S}\psi), \psi(\nu)) \leq \psi(\nu),$$

hence $\Downarrow_- \cdot \mathcal{S} \leq 1_{J_S(X)}$, and therefore $\Downarrow_- \dashv \mathcal{S}$. \square

In general, for a distributor $v : X \multimap X$ and $x \in X$, we consider its local version $v_x : X \multimap 1$ at x defined as $v_x := x^* \circ v$. Observe that for any two $(\mathbb{U}, \mathcal{Q})$ -distributors v, w of type $X \multimap X$, if $v_x \leq w_x$ for all $x \in X$, then $v \leq w$, since $v_x \leq w_x$ iff $v(\nu, x) \leq w(\nu, x)$ for all ν , iff $v \leq w$. Furthermore, we call a $(\mathbb{U}, \mathcal{Q})$ -distributor $v : X \multimap X$ with $v \in J$ *approximating at $x \in X$* if v_x satisfies $X \circ - v_x = x_*$. The counterparts to Proposition 4.4 and Lemma 4.18 read as follows.

Proposition 5.15. *A $(\mathbb{U}, \mathcal{Q})$ -distributor $v : X \multimap X$ is approximating at x , for every $x \in X$, if and only if its mate $\lceil v \rceil$ is of type $X \rightarrow J_S(X)$ and $\mathcal{S} \cdot \lceil v \rceil \cong 1_X$.*

Lemma 5.16. *If $v : X \multimap X$ is approximating at $x \in X$, then $\Downarrow_x \leq v_x$.*

Proof. $\Downarrow_x(\nu) = \bigwedge_{\varphi \in J_S(X)} Q(X(\dot{x}, \mathcal{S}\varphi), \varphi(\nu)) \leq Q(X(\dot{x}, \mathcal{S}v_x), v_x(\nu)) = v_x(\nu)$. \square

Hence, if the way-below distributor exists, it is smaller than any approximation distributor. In particular, \Downarrow is necessarily auxiliary. As in the \mathcal{Q} -case we deduce:

Corollary 5.17. *If \Downarrow exists and is approximation, then \Downarrow is interpolative.*

Lemma 5.18. *Let $v : X \multimap X$ be auxiliary and Scott-continuous. Then, for each $x \in X$, $v_x \leq \Downarrow_x$. Hence, if the way-below distributor \Downarrow exists, then $v \leq \Downarrow$.*

Lemma 5.19. *Let $v : X \multimap X$ be interpolative such that $\mathcal{S}^* \circ v \leq y_*$. Then v is Scott-continuous.*

Of course, $\mathcal{S}^* \circ v \leq y_*$ is equivalent to $v \leq \Downarrow$ assuming that the way-below distributor \Downarrow exists.

Lemma 5.20. *Let $\alpha : X \rightarrow J_S(X)$ be a Scott-continuous $(\mathbb{U}, \mathcal{Q})$ -functor with $\mathcal{S}\alpha \cong 1$. Then $\alpha \dashv \mathcal{S}$.*

Theorem 5.21. *Let $v : X \multimap X \in J$. Then the following are equivalent:*

- (i) $\lceil v \rceil$ is of type $X \rightarrow J_S(X)$ and $\lceil v \rceil \dashv \mathcal{S}$,
- (ii) v is approximating and provides the lifting of \mathcal{S}^* along y_* , i.e. $v = \Downarrow$,
- (iii) v is approximating and Scott-continuous,
- (iv) v is approximating at $x \in X$ for every $x \in X$ and Scott-continuous,
- (v) v is approximating at $x \in X$ for every $x \in X$ and $\lceil v \rceil : X \rightarrow J_S(X)$ is Scott-continuous,
- (vi) for all $\sigma \in UX$ and $\psi \in J_S(X)$ we have $\widehat{X}(U \lceil v \rceil(\sigma), \psi) = X(\sigma, \mathcal{S}(\psi))$.

Theorem 5.22. *The following are equivalent, for a $(\mathbb{U}, \mathcal{Q})$ -category X .*

- (i) X is J -continuous,
- (ii) The way-below $(\mathbb{U}, \mathcal{Q})$ -distributor $\Downarrow : X \multimap X$ exists and is approximating,
- (iii) There exists a Scott-continuous approximating $(\mathbb{U}, \mathcal{Q})$ -distributor $v : X \multimap X$,
- (iv) There exists a Scott-continuous $(\mathbb{U}, \mathcal{Q})$ -distributor $v : X \multimap X$ which is approximating at x , for each $x \in X$.

Finally, we remark that the proofs of Subsection 4.4 can be easily adapted to the $(\mathbb{U}, \mathcal{Q})$ -case, and we have:

Theorem 5.23. *A J -cocomplete $(\mathbb{U}, \mathcal{Q})$ -category X is J -continuous if and only if it is J -projective in $\mathbf{J-Cocont}$.*

Of course, $\mathbf{J-Cocont}$ denotes here the category of J -cocomplete $(\mathbb{U}, \mathcal{Q})$ -categories and Scott-continuous $(\mathbb{U}, \mathcal{Q})$ -functors; and J -projective refers to projectivity with respect to those $(\mathbb{U}, \mathcal{Q})$ -functors in $\mathbf{J-Cocont}$ which have a right inverse in $(\mathbb{U}, \mathcal{Q})\text{-Cat}$.

Example 5.24. We consider the quantale $\mathcal{Q} = \mathbf{2}$. In Example 5.1 we have already seen that this case captures precisely topological spaces and continuous maps. It is shown in [HT08] that a distributor $X \multimap 1$ corresponds to a (possibly improper) filter on the lattice of open subsets of X , and the “presheaf space” \widehat{X} is homeomorphic to the space $F_0(X)$ of all such filters, where the sets

$$\{F \in F_0(X) \mid A \in F\} \quad (A \subseteq X \text{ open})$$

form a basis for the topology on $F_0(X)$ (see also [Esc97]). Note that $F \leq G$ if and only if $F \supseteq G$ in the underlying ordered set $S(F_0(X))$. The Yoneda embedding $\mathbf{y} : X \rightarrow F_0(X)$ sends each point x to the filter $\mathcal{N}(x)$ of all open neighbourhoods of x .

We consider first the absolute case $J = (\mathbb{U}, \mathcal{Q})\text{-Dist}$. A topological space X is cocomplete if and only if $\mathbf{y} : X \rightarrow F_0(X)$ has a left adjoint $\mathcal{S} : F_0(X) \rightarrow X$ in **Top**, which is equivalent to $S(X)^{\text{op}}$ being a continuous lattice. Here \mathcal{S} associates to each filter $F \in F_0(X)$ its smallest convergence point with respect to the order in $S(X)$. Furthermore, the local version \Downarrow_x of the way-below distributor is given by the filter

$$\Downarrow_x = \left\langle \bigcup \{F \in F_0(X) \mid x \leq \mathcal{S}(F)\} \right\rangle$$

generated by $\bigcup \{F \in F_0(X) \mid x \leq \mathcal{S}(F)\}$. A space X is J -continuous if and only if $\Downarrow_- : X \rightarrow F_0(X)$ is continuous and every $x \in X$ is the smallest convergence point of \Downarrow_x . If X is cocomplete, then continuity of $\Downarrow_- : X \rightarrow F_0(X)$ reduces to Scott-continuity of the monotone map $\Downarrow_- : S(X)^{\text{op}} \rightarrow (F_0(X), \subseteq)$ in the usual order-theoretic sense. So far we are not able to give a more elementary description of (absolute) continuity in topological spaces. However, we remark that

- each space of the form $F_0(X)$ is cocomplete and J -continuous,
- and therefore every T_0 -space can be embedded into a cocomplete and J -continuous space.

We finish this paper by mentioning two more examples.

For J being the class of all right adjoint distributors, a topological space X is J -cocomplete if and only if it is weakly sober [CH09], and every topological space is J -continuous.

Further possible choices of J are discussed in [CH08]. For instance, we may consider the class J of all those $(\mathbb{U}, \mathcal{Q})$ -distributors $\varphi : X \multimap Y$ for which $\varphi \circ (-) : \text{Dist}(1, X) \rightarrow \text{Dist}(1, Y)$ preserves certain infima. Note that a distributor $1 \multimap X$ corresponds to a continuous map $X \rightarrow \mathbf{2}$, which in turn corresponds to a closed subset of X . Hence $\text{Dist}(1, X)$ is isomorphic to the lattice of closed subsets of X . In particular, we can chose $J = \{\varphi : X \multimap Y \mid \varphi \circ (-) \text{ preserves the top element}\}$. Then

$$\varphi \in J \iff \forall y \in Y \exists \nu \in UX . \nu \varphi y.$$

Hence, a distributor $\varphi : X \multimap 1$ belongs to J if and only if it corresponds to a proper filter. Therefore

$$\Downarrow_x = \left\langle \bigcup \{F \in F_0(X) \mid x \leq \mathcal{S}(F) \text{ and } F \text{ is proper}\} \right\rangle,$$

and a continuous map $f : X \rightarrow Y$ is J -dense precisely if it is dense in the usual topological sense. Consequently, X is J -cocomplete if and only if X is densely injective. Finally, X is J -continuous if and only if $\Downarrow_- : X \rightarrow F_0(X)$ is continuous and, for every $x \in X$, the filter \Downarrow_x is proper and x is its smallest convergence point.

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