A GENERALIZATION OF THE DUALITY COMPACTNESS THEOREM

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Abstract. In this article we extend the theory of natural dualities for finitary quasivarieties to model categories of finitary limit sketches.

Introduction

The "Duality Compactness Theorem" [11] is a particular result of the theory of natural dualities for finitary quasivarieties (see [4] for a detailed exposition). Subject of this theory is the study of dual adjunctions

$$\mathcal{X} \overset{\varepsilon}{\Rightarrow} \underset{D}{\overset{E}{\rightleftarrows}} \overset{e}{\Leftarrow} \mathcal{A}$$

induced by a pair of objects $(\underline{M}, \tilde{M}) \in \mathsf{Ob}\,\mathcal{A} \times \mathsf{Ob}\,\mathcal{X}$ with the same underlying finite set where

- $\mathcal{A}$ is a finitary quasivariety generated by $\underline{M}$ and
- $\mathcal{X}$ is the category of **Stone**-spaces[1] equipped with finitary (partial) operations and relations generated by $\tilde{M}$.

The goal is to give conditions which guarantee that the dual adjunction above is a dual representation of $\mathcal{A}$ (e is a natural isomorphism) or a dual equivalence between $\mathcal{A}$ and $\mathcal{X}$ (e and ε are natural isomorphisms).

In order to obtain such results, a good strategy is to find conditions which give a duality on the finite level and then "apply some general theory to show that the duality lifts automatically to a duality on the whole of $\mathcal{A}$" [4]. One important result of this general theory is the "Duality Compactness Theorem" (Theorem 2.2.11 in [4]):

> **Theorem.** Let $\mathcal{X}$ be defined by only finitely many (partial) operations and relations and e_A is an isomorphism for each finite algebra A. Then e is a natural isomorphism.

It is well known that each finitary quasivariety is equivalent to a model category of a finitary limit sketch in **Set**. Moreover, the category $\mathcal{X}$ is equivalent to a full subcategory of a model category of a finitary limit sketch in **Stone**. This was the motivation for us to consider a more general situation. Subject of our study are dual adjunctions

$$\mathbf{A} \overset{\eta}{\Rightarrow} \underset{F}{\overset{G}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathbf{B}$$

induced by a pair of objects $(\tilde{A}, \tilde{B}) \in \mathsf{Ob}\,\mathbf{A} \times \mathsf{Ob}\,\mathbf{B}$ with the same underlying finite set where $\mathbf{A}$ is a full subcategory of a model category of a finitary limit sketch $\mathcal{S}_1 = (\mathcal{C}_1, \mathcal{L}_1, \sigma_1)$ in **Stone** generated by $\tilde{A}$ and $\mathbf{B}$ is a full subcategory of model category of a finitary limit sketch $\mathcal{S}_2 = (\mathcal{C}_2, \mathcal{L}_2, \sigma_2)$ in **Set** generated by $\tilde{B}$. The main purpose of this paper is to prove the following generalization of the theorem above.

> **Theorem.** Let $\mathcal{C}_1$ be finitely generated and ε_B is an isomorphism for each finite object $B \in \mathsf{Ob}\,\mathbf{B}$. Then ε is a natural isomorphism.

Besides the larger generality, another advantage of our account is the fact that both categories, $\mathbf{A}$ and $\mathbf{B}$, are defined by the same kind of structure (limit sketches). This makes our results better applicable for the "two-for-one" principle, that is, obtaining a new duality from a given one simply by structure interchange.

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[1]**Stone** denotes here the category of zero-dimensional compact Hausdorff spaces and continuous maps

1. Preliminaries

1.1. *(Natural dualities).* Recall from [8] that a pair of objects $(\tilde{A}, \tilde{B}) \in \mathsf{Ob}\,\mathbf{A} \times \mathsf{Ob}\,\mathbf{B}$ of given concrete categories $(\mathbf{A}, U)$ and $(\mathbf{B}, V)$ over **Set** induces a dual adjunction

$$\mathbf{A} \overset{\eta}{\Rightarrow} \underset{F}{\overset{G}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathbf{B}$$

provided that $U(\tilde{A}) = V(\tilde{B})$ and the conditions

(Init $\mathbf{A}$): For each $A \in \mathsf{Ob}\,\mathbf{A}$, the V-structured source

$$(\mathrm{ev}_{A,a} : \mathsf{hom}(A, \tilde{A}) \to V(\tilde{B}),\ h \mapsto h(a))_{a \in U(A)}$$

admits a V-initial lifting $(\mathrm{ev}_{A,a} : G(A) \to \tilde{B})_{a \in U(A)}$.

(Init $\mathbf{B}$): For each $B \in \mathsf{Ob}\,\mathbf{B}$, the U-structured source

$$(\mathrm{ev}_{B,b} : \mathsf{hom}(B, \tilde{B}) \to U(\tilde{A}),\ h \mapsto h(b))_{b \in V(B)}$$

admits a U-initial lifting $(\mathrm{ev}_{B,a} : F(B) \to \tilde{A})_{b \in V(B)}$.

are fulfilled. We obtain contravariant functors $G : \mathbf{A} \to \mathbf{B}$ and $F : \mathbf{B} \to \mathbf{A}$ as "structured" hom-functors defined by the initial lifts of (Init $\mathbf{A}$) resp. (Init $\mathbf{B}$). The units $\eta : \mathrm{Id}_{\mathbf{A}} \to FG$ and $\varepsilon : \mathrm{Id}_{\mathbf{B}} \to GF$ are given by

$$\eta_A : A \to FG(A),\ a \mapsto \mathrm{ev}_{A,a} \quad \text{and} \quad \varepsilon_B : B \to GF(B),\ b \mapsto \mathrm{ev}_{B,b}$$

for each $A \in \mathsf{Ob}\,\mathbf{A}$ and $B \in \mathsf{Ob}\,\mathbf{B}$. Such a dual adjunction is called *natural dual adjunction induced by* $(\tilde{A}, \tilde{B})$. We can restrict G and F to the full subcategories

$$\begin{aligned} \mathsf{Fix}\,\eta &= \{A \in \mathsf{Ob}\,\mathbf{A} \mid \eta_A \text{ is an isomorphism}\} \text{ and} \\ \mathsf{Fix}\,\varepsilon &= \{B \in \mathsf{Ob}\,\mathbf{B} \mid \varepsilon_B \text{ is an isomorphism}\}, \end{aligned}$$

where they induce a *natural duality.* In general it can be quite difficult to determine these fixed subcategories. However, the fact that η_A is an embedding in $(\mathbf{A}, U)$ can be easily interpreted: η_A is an embedding if and only if $\mathsf{hom}(A, \tilde{A})$ is point separating and U-initial. Of course, an analogous statement holds for ε_B. This suggests the following definition.

Definition 1.1. Let $(\mathbf{A}, U)$ be a concrete category over **Set** and let $\tilde{A} \in \mathsf{Ob}\,\mathbf{A}$. $\tilde{A}$ is called *initial cogenerator* of $(\mathbf{A}, U)$ if, for each $A \in \mathsf{Ob}\,\mathbf{A}$, the source $\mathsf{hom}(A, \tilde{A})$ is point separating and U-initial.

A natural duality between concrete categories $(\mathbf{A}, U)$ and $(\mathbf{B}, V)$ over **Set** induced by $(\tilde{A}, \tilde{B}) \in \mathsf{Ob}\,\mathbf{A} \times \mathsf{Ob}\,\mathbf{B}$ can only exist if $\tilde{A}$ and $\tilde{B}$ are initial cogenerators of $(\mathbf{A}, U)$ and $(\mathbf{B}, V)$ respectively.

We describe now the basic strategy of this paper (see also [2] and [7]). Assume that a given dual adjunction

$$\mathbf{A} \overset{\eta}{\Rightarrow} \underset{F}{\overset{G}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathbf{B}$$

between concrete categories $(\mathbf{A}, U)$ and $(\mathbf{B}, V)$ over **Set** can be restricted to the full subcategories $\mathbf{A}_{\mathrm{fin}}$ and $\mathbf{B}_{\mathrm{fin}}$ of finite objects of $\mathbf{A}$ and $\mathbf{B}$ respectively and, moreover, yields a dual equivalence there. This duality can be extended to the whole of $\mathbf{A}$ and $\mathbf{B}$ provided that every object can be "constructed" from the finite objects and the functors F and G preserve these "constructions". This can be expressed by the following conditions.

(1.) Each object $B \in \mathsf{Ob}\,\mathbf{B}$ is a filtered[2] colimit of finite objects.
(2.) G sends cofiltered limits of finite objects to colimits.
(3.) Each object $A \in \mathsf{Ob}\,\mathbf{A}$ is a cofiltered limit of finite objects.

We remark first that the contravariant functor $F : \mathbf{B} \to \mathbf{A}$, as part of a dual adjunction, sends automatically colimits to limits. This fact together with 2. implies that the endofunctor $GF : \mathbf{B} \to \mathbf{B}$ preserves filtered colimits of finite objects and, dually, $FG : \mathbf{A} \to \mathbf{A}$ preserves

[2] Recall that a diagram scheme I is called *filtered* if each finite subcategory of I (i.e., each subcategory with finitely many morphisms) has a compatible cocone in I.

cofiltered limits of finite objects. Let $B \in \mathsf{Ob}\,\mathbf{B}$ and let $(c_i : B_i \to B)_{i\in I}$ be a presentation of B as filtered colimit of finite objects B_i. We have the following commutative diagram,

$$\begin{array}{ccc} B & \xrightarrow{\varepsilon_B} & GF(B) \\ {\scriptstyle c_i}\uparrow & & \uparrow{\scriptstyle GF(c_i)} \\ B_i & \xrightarrow[\varepsilon_{B_i}]{} & GF(B_i) \end{array}$$

where the left hand side and the right hand side are colimit cones and the ε_{B_i} are isomorphisms. Hence $\varepsilon_B = \mathsf{colim}_{i\in I}\varepsilon_{B_i}$ is an isomorphism. An analogous argument shows that η is a natural isomorphism provided that η_A is an isomorphism for each finite $A \in \mathsf{Ob}\,\mathbf{A}$.

In order to obtain the "Duality Compactness Theorem" in our setting we will follow this idea. For the given dual adjunction we establish first that 1. holds (Proposition 2.1) and give conditions which guarantee that 2. (Lemma 2.2) holds; both together imply our "Duality Compactness Theorem" (Theorem 2.3). Finally, we use a characterization of cofiltered limits of compact Hausdorff spaces to obtain a condition which ensures that 3. holds.

The following proposition reduces 2. to a condition on $\mathsf{hom}(\_, \tilde{A})$ which is often more easily verified. Recall that an object $\tilde{B} \in \mathsf{Ob}\,\mathbf{B}$ of a concrete category $(\mathbf{B}, V)$ over **Set** is *initially dense* in $(\mathbf{B}, V)$ provided that, for each $B \in \mathsf{Ob}\,\mathbf{B}$, the source $\mathsf{hom}(B, \tilde{B})$ is V-initial. Obviously, each initial cogenerator of $(\mathbf{B}, V)$ is initially dense in $(\mathbf{B}, V)$.

Proposition 1.2. *Let* $(\mathbf{A}, U)$ *and* $\mathbf{B}, V)$ *be concrete categories over* **Set** *and*

$$\mathbf{A} \overset{\eta}{\Rightarrow} \underset{F}{\overset{G}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathbf{B}$$

be a natural dual adjunction induced by $(\tilde{A}, \tilde{B})$, *where* $\tilde{B}$ *is initially dense in* $(\mathbf{B}, V)$. *Let* $D : I \to \mathbf{A}$ *be a diagram in* $\mathbf{A}$ *with a concrete limit* $(p_i : L \to D(i))_{i\in I}$ *such that, for each* $i \in I$, $\eta_{D(i)}$ *is an isomorphism. Then* $(G(p_i) : G(L) \to GD(i))_{i\in I}$ *is a colimit of* $GD : I^{\mathrm{op}} \to \mathbf{B}$ *if* $\mathsf{hom}(\_, \tilde{A})$ *sends* $(p_i : L \to D(i))_{i\in I}$ *to a colimit of* $\mathsf{hom}(D(\_), \tilde{A}) : I^{\mathrm{op}} \to \mathbf{Set}$.

Proof. We have to show that the sink $(G(p_i) : GD(i) \to G(L))_{i\in I}$ is V-final, i.e., each map $f : VG(L) \to V(B)$ ($B \in \mathsf{Ob}\,\mathbf{B}$ arbitrary) is actually a $\mathbf{B}$-morphism provided that, for each $i \in I$, $f \circ VG(p_i) = f_i$ is a $\mathbf{B}$-morphism. Since $\tilde{B}$ is initially dense in $(\mathbf{B}, V)$ it is enough to consider a map $f : VG(L) \to V(\tilde{B})$ with codomain $V(\tilde{B})$. For each $i \in I$, since $\eta_{D(i)}$ is an isomorphism, there exists an element $a_i \in UD(i)$ such that $f_i = \mathrm{ev}_{D(i),a_i}$. We define a compatible cone $(\alpha_i : 1 \to UD(i))_{i\in I}$ for UD, hereby $\alpha_i : 1 \to UD(i)$ is given by $0 \mapsto a_i$. The cone $(p_i : U(L) \to UD(i))_{i\in I}$ is a limit of UD, hence there exists an element $a \in U(L)$ such that, for each $i \in I$, $p_i(a) = a_i$. For each $i \in I$ and an arbitrary $h \in \mathsf{hom}(D(i), \tilde{A})$ we have

$$\mathrm{ev}_{L,a} \circ G(p_i)(h) = \mathrm{ev}_{L,a}(h \circ p_i) = h(p_i(a)) = h(a_i) = \mathrm{ev}_{D(i),a_i}(h) = f_i(h),$$

therefore $f = \mathrm{ev}_{L,a}$ and f is a $\mathbf{B}$-morphism. □

1.2. *(Finitary limit sketches).* In this article we will choose the categories $\mathbf{A}$ and $\mathbf{B}$ involved in the dual adjunction as full subcategories of model categories of limit sketches in **Stone** and **Set** respectively. We recall here the basic facts about sketches and locally presentable categories we need and refer for a detailed account of the subject to [1] and [5]. Furthermore, we introduce the concept of single sorted limit sketches.

Definition 1.3. A *finitary limit sketch* is a triple $\mathcal{S} = (\mathcal{C}, \mathcal{L}, \sigma)$ consisting of a small category $\mathcal{C}$, a set $\mathcal{L}$ of diagrams in $\mathcal{C}$ with finite scheme and a function σ which assigns a compatible cone to each diagram of $\mathcal{L}$. A *model* of a finitary limit sketch $\mathcal{S} = (\mathcal{C}, \mathcal{L}, \sigma)$ in a category $\mathbf{K}$ is a functor $F : \mathcal{C} \to \mathbf{K}$ which sends, for each diagram $D : I \to \mathcal{C}$ of $\mathcal{L}$, $\sigma(D)$ to a limit of FD. $\mathsf{Mod}\,(\mathcal{S}, \mathbf{K})$ denotes the full subcategory of $\mathbf{K}^{\mathcal{C}}$ of all models of $\mathcal{S}$ in $\mathbf{K}$.

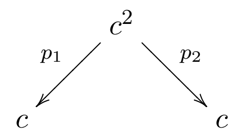
It is not difficult to see that $\mathsf{Mod}\,(\mathcal{S}, \mathbf{K})$ is closed in $\mathbf{K}^{\mathcal{C}}$ under limits – since limits commute with finite limits. Moreover, it is closed under all those colimits which commute in $\mathbf{K}$ with all finite limits. In case $\mathbf{K} = \mathbf{Set}$ these are exactly the filtered colimits.

Examples 1.4. (1) The category of sets equipped with a binary operation and homomorphisms can be expressed as a model category of the following limit sketch $\mathcal{S} = (\mathcal{C}, \mathcal{L}, \sigma)$ in **Set**:

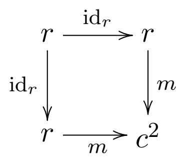
- $\mathcal{C}$ is the category consisting of two objects c and c^2 and has, besides the identity morphisms, three morphisms $o, p_1, p_2 : c^2 \to c$.
- $\mathcal{L}$ contains only the discrete diagram consisting of two copies of c.
- σ assigns the cone $(p_1, p_2 : c^2 \to c)$ to this diagram.

(2) The category of sets equipped with a binary relation and relation-preserving maps is the model category of the following limit sketch $\mathcal{S} = (\mathcal{C}, \mathcal{L}, \sigma)$ in **Set**:

- $\mathcal{C}$ is the category consisting of three objects c, c^2 and r and has, besides the identity morphisms, the morphisms $p_1, p_2 : c^2 \to c$, $m : r \to c^2$ and $p_1 \circ m$ and $p_2 \circ m$.
- $\mathcal{L}$ contains the discrete diagram consisting of two copies of c and the span $r \xrightarrow{m} c^2 \xleftarrow{m} r$.
- σ assigns the cones



and



to these diagrams.

(3) We may add to the category $\mathcal{C}$ of example (2) an arrow $o : r \to c$ and obtain the category of sets equipped with a binary partial operation and homomorphisms as model category of this limit sketch.

Combining all examples above, we are able to sketch categories of sets equipped with a set of (partial) operations and relations. Moreover, we may replace the category **Set** by **Stone** and obtain categories of **Stone**-spaces equipped with continuous (partial) operations and relations as model categories. In case this category can be defined by only finitely many (partial) operations and relations, the underlying category $\mathcal{C}$ of the corresponding sketch is finitely generated[3]. Hence the term "$\tilde{M}$ is of finite type" in the language of [4] translates to "$\mathcal{C}$ is finitely generated" in our setting.

Studying model categories of limit sketches, it is natural to ask how one can describe these categories abstractly. In case we consider model categories in **Set** the answer is provided by the following definition.

Definition 1.5. An object $K \in \mathsf{Ob}\,\mathbf{K}$ of a category $\mathbf{K}$ is called *finitely presentable* if the covariant hom-functor $\mathsf{hom}(K, \_)$ preserves filtered colimits. A category $\mathbf{K}$ is called *locally finitely presentable* provided that the following conditions hold:

(1.) $\mathbf{K}$ is cocomplete.
(2.) There exists, up to isomorphism, only a set of finitely presentable objects in $\mathbf{K}$.
(3.) Each object $K \in \mathsf{Ob}\,\mathbf{K}$ is a filtered colimit of finitely presentable objects.

The model categories of finitary limit sketches in **Set** are precisely (up to equivalence) the locally finitely presentable categories. These categories enjoy many pleasant properties: they are (co)complete and (co)wellpowered and have a generating set. We also remark that each functor between locally finitely presentable categories which preserves limits and filtered colimits has a left adjoint. On the other hand, a characterization of model categories in **Stone** seems to be unknown. However, each model category of a finitary limit sketch in **Stone** is dually equivalent to a model category of a colimit sketch (=the dual concept of limit sketch) in the locally finitely presentable category **Bool** ($\cong$ **Stone**$^{\mathrm{op}}$) and therefore locally copresentable. Hence they are (co)complete and (co)wellpowered as well and have a cogenerating set.

A model category of a limit sketch in a category $\mathbf{K}$ has in general no (canonical) forgetful functor to $\mathbf{K}$. Therefore we introduce the concept of a single sorted limit sketch as a straightforward generalization of single sorted algebraic theories. Let $\mathcal{S} = (\mathcal{C}, \mathcal{L}, \sigma)$ be a finitary limit sketch. Without loss of generality we may assume that $\mathcal{S}$ contains all absolute finite limits.

[3]Recall that a category is finitely generated if there exists a finite set $\mathbb{E}$ of $\mathcal{C}$-morphisms such that each $\mathcal{C}$-morphism is a finite composition of $\mathbb{E}$-morphisms.

The class of all $\mathcal{S}$-*monomorphisms* is defined as the composition closure of the class of all $\mathcal{C}$-morphisms $m : A \to B$ such that the span $A \overset{m}{\to} B \overset{m}{\leftarrow} A$ belongs to L and σ assigns the cone

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle \mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow[m]{} & B \end{array}$$

to this diagram. A $\mathcal{C}$-source $(p_i : C \to C_i)_{i \in I}$ is called $\mathcal{S}$-*limit* if there exists a diagram $D : I \to \mathcal{C}$ in L such that $D(i) = C_i$ for each $i \in I$ and $\sigma(D) = (p_i : C \to C_i)_{i\in I}$. Let $\mathcal{C}_0 \subset \mathcal{C}$ be a full subcategory of $\mathcal{C}$. We define the following full subcategories of $\mathcal{C}$:

$$\begin{aligned} \mathsf{Sub}_{\mathcal{S}}(\mathcal{C}_0) &= \{C \in \mathsf{Ob}\,\mathcal{C} \mid \text{there exists a } \mathcal{S}\text{-monomorphism } m : C \to C_0 \text{ such that} \\ &\quad\; C_0 \in \mathsf{Ob}\,\mathcal{C}_0\}, \\ \mathsf{Lim}_{\mathcal{S}}(\mathcal{C}_0) &= \{C \in \mathsf{Ob}\,\mathcal{C} \mid \text{there exists a } \mathcal{S}\text{-limit } (p_i : C \to C_i)_{i\in I} \text{ such that} \\ &\quad\; C_i \in \mathsf{Ob}\,\mathcal{C}_0 \text{ for each } i \in I\}. \end{aligned}$$

Let $C \in \mathsf{Ob}\,\mathcal{C}$. Inductively we define a chain $\mathcal{G}_n(C)$ $(n \in \mathbb{N})$ of full subcategories of $\mathcal{C}$ in the following way:

(1.) We put $\mathcal{G}_0(C) = \{C\}$ and,
(2.) for each $n \geq 0$, $\mathcal{G}_{n+1}(C) = \mathsf{Sub}_{\mathcal{S}}(\mathsf{Lim}_{\mathcal{S}}(\mathcal{G}_n(C)))$.

Definition 1.6. Let $\mathcal{S} = (\mathcal{C}, \mathcal{L}, \sigma)$ be a finitary limit sketch.

(1.) An object $C_0 \in \mathsf{Ob}\,\mathcal{C}$ is called *sketch-cogenerator* of $\mathcal{S}$ if $\mathcal{C} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n(C_0)$.
(2.) $\mathcal{S}$ is called *single sorted* provided that there exists a sketch-cogenerator C_0 of $\mathcal{S}$.

Of course, each single sorted finitary algebraic theory is single sorted in the sense above. Moreover, each sketch of Example 1.4 is single sorted.

Lemma 1.7. *Let $\mathcal{S} = (\mathcal{C}, \mathcal{L}, \sigma)$ be a finitary, single sorted limit sketch with sketch-cogenerator C_0. For each $C \in \mathsf{Ob}\,\mathcal{C}$, there exists a finite subset $M \subset \mathsf{hom}(C, C_0)$ such that, for each model $F : \mathcal{C} \to \mathbf{K}$ of $\mathcal{S}$, the source $(F(f) : F(C) \to F(C_0))_{f\in M}$ is a mono-source in $\mathbf{K}$.*

Corollary 1.8. *Let $\mathcal{S} = (\mathcal{C}, \mathcal{L}, \sigma)$ be a finitary, single sorted limit sketch with sketch-cogenerator C_0.*

(1.) *For each category $\mathbf{K}$, the evaluation functor $\mathrm{Ev}_{C_0} : \mathsf{Mod}\,(\mathcal{S}, \mathbf{K}) \to \mathbf{K}$ is faithful.*
(2.) *Let $(\mathbf{K}, U)$ be a concrete category over $\mathbf{Set}$ such that U preserves finite mono-sources. Let $F : \mathcal{C} \to \mathbf{K}$ be any model of $\mathcal{S}$ in $\mathbf{K}$. Then $UF(C)$ is finite for each $C \in \mathsf{Ob}\,\mathcal{C}$ if and only if $UF(C_0)$ is finite.*

2. Natural dualities between model categories of limit sketches

We describe first our *basic situation* which we assume to be given throughout this section. Let $\mathcal{S}_1 = (\mathcal{C}_1, \mathcal{L}_1, \sigma_1)$ and $\mathcal{S}_2 = (\mathcal{C}_2, \mathcal{L}_2, \sigma_2)$ be single sorted, finitary limit sketches with sketch-cogenerators C_1 and C_2 respectively. We obtain concrete categories

$$(\mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone}), |\_| \circ \mathrm{Ev}_{C_1}) \quad \text{and} \quad (\mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set}), \mathrm{Ev}_{C_2})$$

over $\mathbf{Set}$, where $|\_| : \mathbf{Stone} \to \mathbf{Set}$ denotes the canonical forgetful functor. We assume that objects $\tilde{A} \in \mathsf{Ob}\,\mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone})$ and $\tilde{B} \in \mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set})$ with finite underlying set $|\tilde{A}(C_1)| = \tilde{B}(C_2)$ are given.

The category $\mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set})$ is, as a locally finitely presentable category, (co)complete and (co)wellpowered and the forgetful functor $\mathrm{Ev}_{C_2} : \mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set}) \to \mathbf{Set}$ has a left adjoint and preserves filtered colimits. The category $\mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone})$ is locally copresentable and therefore (co)complete and (co)wellpowered and has a cogenerating set. Hence the functor $\mathrm{Ev}_{C_1} : \mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone}) \to \mathbf{Stone}$ has a left adjoint as well.

In general, the objects $\tilde{A}$ and $\tilde{B}$ will not be initial cogenerators of $\mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone})$ resp. $\mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set})$. Hence we must restrict our attention to the full subcategories of those objects C of $\mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone})$ and $\mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set})$ respectively for which the sources $\mathsf{hom}(C, \tilde{A})$ resp. $\mathsf{hom}(C, \tilde{B})$ are point separating and initial. In order to cover the results presented in [4], we consider the following more general situation: Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be classes of $\mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone})$- resp. $\mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set})$-morphisms closed under composition, pullback and intersection stable, containing all regular monomorphisms and contained in the class of all embeddings. Of course, the leading example we have in mind is to choose $\mathbb{M}_1$ and $\mathbb{M}_2$ as the class of all embeddings. Another possible choice is the inclusion of substructures of [4]. We remark that the name "embedding" used in [4] for these morphisms is misleading since they form in general only a proper subclass of the class of all embeddings[4]. $\mathbb{M}_1$ is part of a factorization system ($\mathbb{M}_1$-**ExtrEpi**,$\mathbb{M}_1$) on $\mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone})$, where $\mathbb{M}_1$-**ExtrEpi** denotes the class of all $\mathbb{M}_1$-*extremal epimorphisms* which are exactly those morphisms f of $\mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone})$ satisfying the following condition: whenever $f = m \circ g$, where $m \in \mathbb{M}_1$, then m must be an isomorphism. Analogously, $\mathbb{M}_2$ is part of a factorization system ($\mathbb{M}_2$-**ExtrEpi**,$\mathbb{M}_2$) on $\mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set})$. We define $\mathbf{A}$ as the full subcategory of $\mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone})$ of all $\mathbb{M}_1$-subobjects of powers of $\tilde{A}$. $\mathbf{B}$ denotes the full subcategory of $\mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set})$ of all $\mathbb{M}_2$-subobjects of powers of $\tilde{B}$. $\mathbf{A}$ is an $\mathbb{M}_1$-**ExtrEpi**-reflective subcategory of $\mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone})$ with left adjoint $R_{\tilde{A}} : \mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone}) \to \mathbf{A}$ and $\mathbf{B}$ is an $\mathbb{M}_2$-**ExtrEpi**-reflective subcategory of $\mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set})$ with left adjoint $R_{\tilde{B}} : \mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set}) \to \mathbf{B}$. Let $U : \mathbf{A} \to \mathbf{Set}$ denote the restriction of $|\_| \circ \mathrm{Ev}_{C_1}$ to $\mathbf{A}$ and let $U^* : \mathbf{A} \to \mathbf{Stone}$ denote the restriction of Ev_{C_1} to $\mathbf{A}$. Note that a point separating source in $\mathbf{A}$ is U-initial if and only if it is U^*-initial. Let $V : \mathbf{B} \to \mathbf{Set}$ denote the restriction of Ev_{C_2} to $\mathbf{B}$. Obviously, $\tilde{A}$ and $\tilde{B}$ are initial cogenerators of $(\mathbf{A}, U)$ and $(\mathbf{B}, V)$ respectively.

Finally, we assume that $(\tilde{A}, \tilde{B})$ induces a natural dual adjunction

$$\mathbf{A} \overset{\eta}{\Rightarrow} \underset{F}{\overset{G}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathbf{B} \tag{1}$$

between $(\mathbf{A}, U)$ and $(\mathbf{B}, V)$. We are now going to present conditions which guarantee that (1) is already a dual equivalence provided that its restriction to the full subcategories $\mathbf{A}_{\mathsf{fin}}$ and $\mathbf{B}_{\mathsf{fin}}$ of finite objects of $\mathbf{A}$ and $\mathbf{B}$ respectively is.

Our first goal is to prove that each $\mathbf{B}$-object is a filtered colimit of finite objects.

Proposition 2.1. *For each finitely presentable object $P \in \mathsf{Ob}\,\mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set})$, the reflection $R_{\tilde{B}}(P)$ is finite. Hence each $B \in \mathsf{Ob}\,\mathbf{B}$ is a filtered colimit of finite objects in $\mathbf{B}$.*

Proof. Obviously, an object $B \in \mathsf{Ob}\,\mathbf{B}$ is finite if and only if $\mathsf{hom}(B, \tilde{B})$ is finite. Recall that the inclusion functor $\mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set}) \hookrightarrow \mathbf{Set}^{\mathcal{C}_2}$ is right adjoint and preserves filtered colimits, hence the reflection $\chi(C)$ ($C \in \mathsf{Ob}\,\mathcal{C}_2$) of the hom-functor $\mathsf{hom}(C, \_)$ is finitely presentable in $\mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set})$. We assume first that $P = \chi(C)$ for some $C \in \mathsf{Ob}\,\mathcal{C}_2$. We have

$$\mathsf{Nat}(R_{\tilde{B}}(\chi(C)), \tilde{B}) \cong \mathsf{Nat}(\chi(C), \tilde{B}) \cong \mathsf{Nat}(\mathsf{hom}(C, \_), \tilde{B}) \cong \tilde{B}(C),$$

hence $R_{\tilde{B}}(\chi(C))$ is finite. The general case follows now from the fact that the full subcategory of all finitely presentable objects of $\mathsf{Mod}\,(\mathcal{S}_2, \mathbf{Set})$ is the closure of $\{\chi(C) \,|\, C \in \mathsf{Ob}\,\mathcal{C}_2\}$ under finite colimits and that the finite colimit of finite objects in $\mathbf{B}$ is finite. $\square$

Next we give conditions which guarantee that the contravariant functor $G : \mathbf{A} \to \mathbf{B}$ sends cofiltered limits of finite objects to filtered colimits. According to Proposition 1.2 it is sufficient to show that $\tilde{A}$ is finitely copresentable in $\mathbf{A}$. Note that, since $\mathbf{A}$ is a reflective subcategory of $\mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone})$, an $\mathbf{A}$-object is finitely copresentable in $\mathbf{A}$ if it is in $\mathsf{Mod}\,(\mathcal{S}_1, \mathbf{Stone})$.

Lemma 2.2. *Let $\mathcal{S} = (\mathcal{C}, \mathcal{L}, \sigma)$ be a finitary limit sketch such that $\mathcal{C}$ is finitely generated. An object $M \in \mathsf{Ob}\,\mathsf{Mod}\,(\mathcal{S}, \mathbf{Stone})$ is finitely copresentable provided that, for each $C \in \mathsf{Ob}\,\mathcal{C}$, $M(C)$ is a finite discrete space.*

Proof. Let $\mathbb{E}$ be a finite subset of $\mathsf{Mor}\,\mathcal{C}$ such that each $\mathcal{C}$-morphism is a finite composition of $\mathbb{E}$-morphisms. In particular, $\mathcal{C}$ has only finitely many objects. There is nothing to prove if $\mathcal{C}$ is the empty category, hence we may assume that $\mathsf{Mor}\,\mathcal{C}$ and hence $\mathbb{E}$ are non-empty.

Let $D : I \to \mathsf{Mod}\,(\mathcal{S}, \mathbf{Stone})$ be a cofiltered diagram, let $(p_i : F \to D(i))_{i \in I}$ be a limit of D and let $\eta : F \to M$ be a natural transformation. Let C be any $\mathcal{C}$-object. $M(C)$ is, as a finite space, finitely copresentable in $\mathbf{Stone}$ and $((p_i)_C : F(C) \to D(i)(C))_{i \in I}$ is a limit of $\mathrm{Ev}_C \circ D$

[4]Consider, in the category of spaces equipped with a partial binary operation, the three-elemented discrete space $\{1, 2, 3\}$ where only $1 + 2 = 3$ is defined. The inclusion of the two-elemented subspace $\{1, 2\}$, where the domain of the operation is empty, is an embedding but not a substructure.

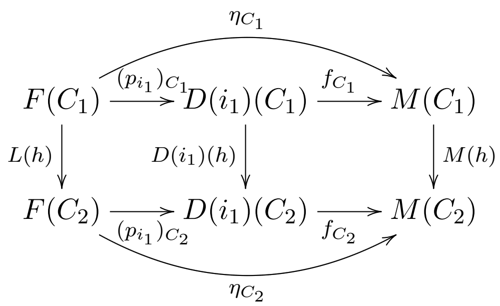
in **Stone**, hence we can find an I-object i_C and a continuous map $f_C^{\#} : D(i_C)(C) \to M(C)$ such that $f_C^{\#} \circ (p_{i_C})_C = \eta_C$. Since $\mathcal{C}$ has only finitely many objects and I is cofiltered, there exists a cone $(k_C : i_1 \to i_C)_{C \in \mathsf{Ob}\,\mathcal{C}}$ in I. We put

$$f_C = f_C^{\#} \circ D(k_C)_C$$

for each $C \in \mathsf{Ob}\,\mathcal{C}$, it holds

$$f_C \circ (p_{i_1})_C = f_C^{\#} \circ D(k_C)_C \circ (p_{i_1})_C = f_C^{\#} \circ (p_{i_C})_C = \eta_C.$$

In general, the family $(f_C)_{C \in \mathsf{Ob}\,\mathcal{C}}$ fails to be a natural transformation. Let $h : C_1 \to C_2$ be any $\mathbb{E}$-morphism. In the diagram



the outer and the left hand square commute, hence we have

$$M(h) \circ f_{C_1} \circ (p_{i_1})_{C_1} \quad = \quad f_{C_2} \circ D(i_1)(h) \circ (p_{i_1})_{C_1}.$$

$M(C_2)$ is finitely copresentable in **Stone**, hence there exists an I-morphism $l_h : i_h \to i_1$ such that

$$M(h) \circ f_{C_1} \circ D(l_h)_{C_1} = f_{C_2} \circ D(i_1)(h) \circ D(l_h)_{C_1}.$$

Since $\mathbb{E}$ is finite, there exists a cone $(k_h : i_0 \to i_h)_{h \in \mathbb{E}}$ in I such that, for all $h_1, h_2 \in \mathbb{E}$, $l_{h_1} \circ k_{h_1} = l_{h_2} \circ k_{h_2} = k$. We put $\alpha_C = f_C \circ D(k)_C$. Since the equations

$$\begin{aligned} \alpha_C \circ (p_{i_0})_C &= f_C \circ D(k)_C \circ (p_{i_0})_C \\ &= f_C \circ (p_{i_1})_C \\ &= \eta_C \end{aligned}$$

for each $C \in \mathsf{Ob}\,\mathcal{C}$ and

$$\begin{aligned} \alpha_{C_2} \circ D(i_0)(h) &= f_{C_2} \circ D(k)_{C_2} \circ D(i_0)(h) \\ &= f_{C_2} \circ D(i_1)(h) \circ D(k)_{C_1} \\ &= f_{C_2} \circ D(i_1)(h) \circ D(l_h)_{C_1} \circ D(k_h)_{C_1} \\ &= M(h) \circ f_{C_1} \circ D(l_h)_{C_1} \circ D(k_h)_{C_1} \\ &= M(h) \circ f_{C_1} \circ D(k)_{C_1} \\ &= M(h) \circ \alpha_{C_1} \end{aligned}$$

for each $h \in \mathbb{E}$ hold and each $\mathcal{C}$-morphism is a finite composition of $\mathbb{E}$-morphisms, we conclude that the family $\alpha = (\alpha_C)_{C \in \mathsf{Ob}\,\mathcal{C}}$ is a natural transformation $\alpha : D(i_0) \to M$ such that $\alpha \circ p_{i_0} = \eta$.

It remains to show that the factorization is essentially unique. Let $i_0 \in I$ and

$$\alpha = (\alpha_C)_{C \in \mathsf{Ob}\,\mathcal{C}} : D(i_0) \to M \quad \text{and} \quad \beta = (\beta_C)_{C \in \mathsf{Ob}\,\mathcal{C}} : D(i_0) \to M$$

be natural transformations such that $\alpha \circ p_{i_0} = \eta = \beta \circ p_{i_0}$. In particular, for each $C \in \mathsf{Ob}\,\mathcal{C}$ we have

$$\alpha_C \circ (p_{i_0})_C = \eta_C \quad \text{and} \quad \beta_C \circ (p_{i_0})_C = \eta_C,$$

hence there exists an I-morphism $h_C : i_C \to i_0$ such that

$$\alpha_C \circ D(h_C)_C \quad = \quad \beta_C \circ D(h_C)_C.$$

Let $(k_C : i_1 \to i_C)_{C\in \mathsf{Ob}\,\mathcal{C}}$ be a family of I-morphisms with $h_{C_2} \circ k_{C_2} = h_{C_2} \circ k_{C_2} = k$ for all $C_1, C_2 \in \mathsf{Ob}\,\mathcal{C}$. We have

$$\begin{aligned} \alpha_C \circ D(k)_C &= \alpha_C \circ D(h_C)_C \circ D(k_C)_C \\ &= \beta_C \circ D(h_C)_C \circ D(k_C)_C \\ &= \beta_C \circ D(k)_C \end{aligned}$$

for each $C \in \mathsf{Ob}\,\mathcal{C}$. □

As a consequence of Proposition 2.1 and Lemma 2.2 we obtain the promised generalization of the "Duality Compactness Theorem".

Theorem 2.3. *Given the basic situation where, in addition, $\mathcal{C}_1$ is finitely generated and ε_B is an isomorphism for each finite $B \in \mathsf{Ob}\,\mathbf{B}$. Then ε is a natural isomorphism.*

It remains to give conditions which guarantee that each $\mathbf{A}$-object is a cofiltered limit of finite objects. To do this, we will use the following characterization of cofiltered limits in $\mathbf{Comp}_2$ (see [3]), the category of compact Hausdorff spaces and continuous maps.

Theorem 2.4. *Let $D : I \to \mathbf{Comp}_2$ be a cofiltered diagram and let $(p_i : L \to D(i))_{i\in I}$ be a compatible cone for D. The following assertions are equivalent.*

(1.) *$(p_i : L \to D(i))_{i\in I}$ is a limit of D.*
(2.) *The following two conditions are fulfilled.*
 (a) *$(p_i : L \to D(i))_{i\in I}$ is point separating.*
 (b) *For each $i \in I$,*

$$\mathsf{Im} p_i = \bigcap_{j \xrightarrow{k} i} \mathsf{Im} D(k).$$

The subcategory **Stone** is closed in $\mathbf{Comp}_2$ under limits, therefore this characterization holds for cofiltered limits in **Stone** as well. For each object $A \in \mathsf{Ob}\,\mathbf{A}$, the canonical diagram $D_A : A/\mathbf{A}_{\text{fin}} \to \mathbf{A}$ of A with respect to the (small) full subcategory $\mathbf{A}_{\text{fin}} \subset \mathbf{A}$ of all finite objects of $(\mathbf{A}, U)$ is cofiltered and the cone

$$(f : A \to E)_{f\in A/\mathbf{A}_{\text{fin}}}$$

is compatible for D_A. This cone is a limit cone for D_A if and only if it is

(1.) U^*-initial and point separating and
(2.) the underlying cone $(U^*(f) : U^*(A) \to U^*(E))_{f\in A/\mathbf{A}_{\text{fin}}}$ is a limit of the underlying diagram $U^* D_A : A/\mathbf{A}_{\text{fin}} \to \mathbf{Stone}$.

The source $(f : A \to E)_{f\in A/\mathbf{A}_{\text{fin}}}$ contains the source $\mathsf{hom}(A, \tilde{A})$, therefore 1. holds. According to Theorem 2.4, 2. holds if and only if

(*) for each $\mathbf{A}$-morphism $f : A \to E$ with finite codomain E and each $x \in U(E) - \mathsf{Im} f$, there exist a finite object $E' \in \mathsf{Ob}\,\mathbf{A}$ and $\mathbf{A}$-morphisms $f' : A \to E'$ and $e : E' \to E$ such that $f = e \circ f'$ and $x \notin \mathsf{Im} e$.

The condition (*) is for instance fulfilled if $(\mathbf{A}, U)$ has (**Surj**,**Inj**)-factorizations. Putting everything together we obtain our main result:

Theorem 2.5. *Assume that the restriction of the given dual adjunction (1) to $\mathbf{A}_{\text{fin}}$ and $\mathbf{B}_{\text{fin}}$ is a dual equivalence. Then (1) is a dual equivalence provided that the following hold:*

(1.) *$\mathcal{C}_1$ is finitely generated and*
(2.) *$(\mathbf{A}, U)$ has (**Surj**,**Inj**)-factorizations.*

Theorem 2.5 can be used to prove new duality theorems from given ones just by structure interchange, as illustrated by the following simple example. The Priestley Duality Theorem (see [9] and [10]) states that the 2-chain induces a natural dual equivalence between the category of Priestley spaces (i.e., the full subcategory of **StonePos** consisting of those partially ordered **Stone**-spaces A such that the source $\mathsf{hom}(A, 2)$ is point separating and initial) and the category $\mathbf{DLat}_{0,1}$ of bounded distributive lattices. Now we interchange the structure and consider the categories $\mathbf{StoneDLat}_{0,1}$ and **Pos**. It is well-known that the 2-chain is an initial cogenerator in **Pos** and $\mathbf{StoneDLat}_{0,1}$ (for the second case, see [7], VI, 2) and, moreover, induces a natural dual adjunction between these categories. In fact, on the full subcategories of finite objects we obtain a duality, thanks to the (finite version of the) Priestley Duality Theorem. Theorem 2.5 implies that $\mathbf{StoneDLat}_{0,1}$ and **Pos** are actually dually equivalent. Note that this example is outside the scope of the results of [4] since **Pos** does not have a dense set of regular projectives (see [1]) and hence is not a quasivariety.

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