A DUALITY OF QUANTALE-ENRICHED CATEGORIES

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Abstract. We describe a duality for quantale-enriched categories that extends the Lawson duality for continuous dcpos: for any saturated class J of modules that commute with certain weighted limits, and under an appropriate choice of morphisms, the category of J-cocomplete and J-continuous quantale-enriched categories is self-dual.

1. Introduction

In [14] we observed that the left adjoint to the left adjoint to the Yoneda embedding in a quantale-enriched category X can be interpreted as a notion of approximation in X. Thus in directed-complete posets, approximation is the way-below relation [13].I.1.; in complete lattices the totally-below relation [24]; and in (generalised) metric spaces a distance $\Downarrow\colon X \times X \to [0,\infty]$ such that every $x \in X$ is a "metric supremum" of $\Downarrow(-,x)$ [14].

The purpose of this paper is to develop a duality theory for $\mathcal{Q}$-categories that extends the Lawson duality for continuous dcpos [22]. Recall that Lawson's theorem states that the category of continuous dcpos with Scott-open filter reflecting maps is self-dual. We show that under an appropriate choice of morphisms the category of J-cocomplete and J-continuous (= admitting approximation) $\mathcal{Q}$-categories is self-dual. Our duality theorem holds for any saturated class J of modules that preserve certain limits; therefore it works uniformly for continuous domains, completely distributive complete lattices, Yoneda-complete quasi-metric spaces, totally distributive $\mathcal{Q}$-categories, and perhaps many other familiar structures from the borderline of metric and order theory. We also answer an open question of [31]: in Example 4.1 we prove that flat modules do not correspond to the points of the forward Cauchy net completion in ultrametric spaces. Finally, in Subsection 4.5 we present a duality which is different than any other in this paper since it does not follow from the main theorem 3.9. This example demonstrates the potential of extending our duality beyond the class of limit weights.

Our feet rest on shoulders of many. Hausdorff's point of view that a metric is a relation valued in non-negative real numbers, brought to light by [23], led to a development of an unified categorical/algebraic description of topology, uniformity, order and metric [5, 7, 6]. The idea of relative cocompleteness was developed in [16, 1, 19, 18, 17, 27]. Our primary examples of classes of modules have already been studied in [11, 27, 31]. We do hope that our results will be of interest to those who research extensions of dualities in domain theory (e.g. the pointfree

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setting of preframes [9]) and to those who work with categories where the left adjoint to Yoneda embedding has a left adjoint; research in this direction include: [15, 20, 10, 26, 30].

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Note from Dirk Hofmann: Paweł Waszkiewicz died in a car accident in August 2011. I lost a great friend and collaborator.

2. Preliminaries

In this section we recall some basic facts about $\mathcal{Q}$-categories, $\mathcal{Q}$-functors and $\mathcal{Q}$-modules, for a commutative unital quantale $\mathcal{Q}$. A very nice introduction into this topic can be found in Lawvere's ground-breaking paper [23], and, if not stated otherwise, we refer for details to [23].

2.1. **Quantales.** A $\mathcal{Q} = (\mathcal{Q}, \otimes, \mathbf{1})$ is a commutative unital quantale (in short: a quantale), that is, $\mathcal{Q}$ is a complete lattice with a commutative binary operation $\otimes$ and neutral element $\mathbf{1}$, such that $u \otimes (-)$ preserves suprema, for all $u \in \mathcal{Q}$. Consequently, $\mathcal{Q}$ has an "internal hom" $\mathcal{Q}(-,-)$, characterized by

$$z \leq \mathcal{Q}(u,v) \iff z \otimes u \leq v,$$

for all $z, u, v \in \mathcal{Q}$. We also assume that $\perp \neq \mathbf{1}$. Examples of quantales include:

Example 2.1. The two element lattice $\mathbf{2} = (\{\perp, \mathbf{1}\}, \leqslant, \wedge, \mathbf{1})$.

Example 2.2. The extended real half line $[0, \infty]$ in the order opposite to the natural one, with addition as tensor.

Example 2.3. Every Heyting algebra with infimum as tensor is a quantale.

Example 2.4. Let $(M, *, e)$ be a commutative monoid. The powerset lattice $(\mathcal{P}(M), \subseteq)$ with tensor given by

$$A \otimes B = \{a * b \mid a \in A \text{ and } b \in B\}$$

and the unit $\{e\}$ is a quantale.

Example 2.5. From [12]. Let $\mathcal{M}$ be the set of monotone maps from $[0, \infty]$ to $[0, 1]$. Let Δ be the subset of $\mathcal{M}$ of all functions f such that for all $x \in [0, \infty]$, $f(x) = \sup_{y<x} f(y)$, considered with the pointwise ordering. Then Δ is a quantale in various ways, for instance with tensor

$$(f \otimes g)(x) = \sup_{u+v \leqslant x}(f(u) + g(v) - 1),$$

or with tensor

$$(f \otimes g)(x) = \sup_{u+v \leqslant x}(f(u) \cdot g(v)),$$

and in both cases the unit is

$$\mathbf{1}(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise.} \end{cases}$$

It is worth mentioning that Δ-categories (see below) are so-called probabilistic metric spaces [28].

2.2. **$\mathcal{Q}$-categories.** We recall that a category enriched in a quantale $\mathcal{Q}$, for short: a *$\mathcal{Q}$-category*, is a set X with a map $X\colon X\times X\to\mathcal{Q}$, called *the structure of* X, with two properties: $\mathbf{1}\leqslant X(x,x)$ for all $x\in X$ (reflexivity), and $X(x,y)\otimes X(y,z)\leqslant X(x,z)$ for all $x,y,z\in X$ (transitivity). In our paper $\mathcal{Q}$-**Cat** denotes the category of $\mathcal{Q}$-categories, where morphisms, called *$\mathcal{Q}$-functors*, are maps $f\colon X\to Y$ such that $X(x,z)\leqslant Y(fx,fz)$ for all $x,z\in X$. For example $\mathbf{Met}:=[0,\infty]$-**Cat** is Lawvere's category of generalised metric spaces [23], where reflexivity and transitivity correspond respectively to the assumption of self-distance being zero and to the triangle inequality. As another example we consider **2-Cat**, which is isomorphic to the category of preordered sets and monotone maps, and will henceforth be denoted by **Ord**. The quantale $\mathcal{Q}$ is made into a $\mathcal{Q}$-category by its internal hom $\mathcal{Q}(-,-)$. By X^{op} we mean the $\mathcal{Q}$-category dual to X, that is, the underlying set of X^{op} is X and the structure is $X^{\mathrm{op}}(x,y)=X(y,x)$.

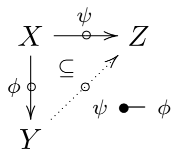
The tensor product on $\mathcal{Q}$ gives rise to a tensor product $X\otimes Y$ on $\mathcal{Q}$-**Cat**; here the underlying set of $X\otimes Y$ is the Cartesian product $X\times Y$, and its structure is given by $X\otimes Y((x,y),(z,w))=X(x,z)\otimes Y(y,w)$. This operation is in general different from the categorical product $X\times Y$ in the category $\mathcal{Q}$-**Cat** whose structure is $X\times Y((x,y),(z,w))=X(x,z)\wedge Y(y,w)$. More importantly, it is better behaved than the categorical product since it makes $\mathcal{Q}$-**Cat** a monoidal closed category, with internal hom Y^X being the set of all $\mathcal{Q}$-functors of type $X\to Y$ considered with the structure $Y^X(f,g):=\bigwedge_{x\in X}Y(fx,gx)$. Since tensor is left adjoint to internal hom, every $\mathcal{Q}$-functor $g\colon X\otimes Y\to Z$ has its *exponential mate* $\ulcorner g\urcorner\colon Y\to Z^X$. We will be particularly interested in the case $Y=\mathcal{Q}$, and write $\widehat{X}$ as a shorthand for $\mathcal{Q}^{X^{\mathrm{op}}}$ and $[-,-]$ to denote its structure. It is worth noting that the structure of X is always a $\mathcal{Q}$-functor of type $X^{\mathrm{op}}\otimes X\to\mathcal{Q}$, and its exponential mate is the *Yoneda embedding* $\mathsf{y}_X: X\to\widehat{X}$. Furthermore, for all $x\in X$ and $f\in\widehat{X}$, we have $\widehat{X}(\mathsf{y}_Xx,f)=fx$, and this equality is the statement of the Yoneda Lemma for $\mathcal{Q}$-categories. It implies in particular that the Yoneda embedding is fully faithful, justifying therefore the designation "embedding".

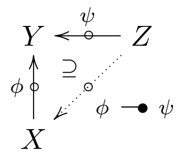
2.3. **$\mathcal{Q}$-modules.** Besides $\mathcal{Q}$-functors, there is another important type of morphisms between $\mathcal{Q}$-categories: $\mathcal{Q}$-modules. Here a *$\mathcal{Q}$-module* (or plainly: a *module*) is a $\mathcal{Q}$-functor of type $X^{\mathrm{op}}\otimes Y\to\mathcal{Q}$. We observed already above that the structure of any $\mathcal{Q}$-category X is a module; moreover, any two modules $\phi\colon X^{\mathrm{op}}\otimes Y\to Q$ and $\psi\colon Y^{\mathrm{op}}\otimes Z\to Q$ can be composed to give a module of type $X^{\mathrm{op}}\otimes Z\to Q$:

$$(\psi\cdot\phi)(x,z):=\bigvee_{y\in Y}(\phi(x,y)\otimes\psi(y,z)).$$

Therefore we think of $\phi\colon X^{\mathrm{op}}\otimes Y\to Q$ as an arrow $\phi\colon X\,{\circ\!\!\to}\,Y$, which, by the above, can be composed with $\psi\colon Y\,{\circ\!\!\to}\,Z$ to give $\psi\cdot\phi\colon X\,{\circ\!\!\to}\,Z$. Since also $Y\cdot\phi=\phi=\phi\cdot X$ for every $\mathcal{Q}$-module $\phi\colon X\,{\circ\!\!\to}\,Y$, $\mathcal{Q}$-categories and $\mathcal{Q}$-modules form a category, denoted as $\mathcal{Q}$-**Mod**, with composition defined as above and the identity on a $\mathcal{Q}$-category X is $X\colon X\,{\circ\!\!\to}\,X$.

The set of all modules of type $X\,{\circ\!\!\to}\,Y$ becomes a complete lattice via the pointwise order where the supremum ϕ of a family $\phi_i\colon X\,{\circ\!\!\to}\,Y$ $(i\in I)$ of modules can be calculated as $\phi(x,y)=\bigvee_{i\in I}\phi_i(x,y)$. Furthermore, composition of modules preserves this suprema on both sides, and therefore the maps $-\cdot\phi$ and $\phi\cdot -$ have right adjoints $-\bullet\!\!-\,\phi$ and $\phi\,-\!\!\bullet\,-$ respectively. By definition, for a $\mathcal{Q}$-module $\phi: X\,{\circ\!\!\to}\,Y$, a right adjoint to $-\cdot\varphi:\mathcal{Q}\text{-}\mathbf{Mod}(Y,Z)\to\mathcal{Q}\text{-}\mathbf{Mod}(X,Z)$ must give, for each $\psi: X\,{\circ\!\!\to}\,Z$, the largest $\mathcal{Q}$-module of type $Y\,{\circ\!\!\to}\,Z$ whose composite with φ is contained in ψ, and a right adjoint to $\varphi\cdot -:\mathcal{Q}\text{-}\mathbf{Mod}(Z,X)\to\mathcal{Q}\text{-}\mathbf{Mod}(Z,Y)$ must provide, for each $\psi: Z\,{\circ\!\!\to}\,Y$, the largest $\mathcal{Q}$-module of type $Z\,{\circ\!\!\to}\,X$ whose composite with φ is contained in ψ.





The $\mathcal{Q}$-module $\psi \bullet\!\!- \phi$ is called the *extension* of ψ along ϕ, and $\phi -\!\!\bullet \psi$ the *lifting* of ψ along ϕ. Explicitly, given $\phi\colon X \mathrel{-\!\!\circ\!\!\to} Y$,

$$(\psi \bullet\!\!- \phi)(y,z) = \bigwedge_{x\in X} \mathcal{Q}(\phi(x,y),\psi(x,z))$$

for any $\psi\colon X \mathrel{-\!\!\circ\!\!\to} Z$, and

$$(\phi -\!\!\bullet \psi)(z,x) = \bigwedge_{y\in Y} \mathcal{Q}(\phi(x,y),\psi(z,y))$$

for any $\psi\colon Z \mathrel{-\!\!\circ\!\!\to} Y$. This construction will be used to define the so-called *way-below module* in Section 2.5.

Any $\mathcal{Q}$-functor $f\colon X \to Y$ gives rise to two $\mathcal{Q}$-modules, namely $f_*\colon X \mathrel{-\!\!\circ\!\!\to} Y$, $f_*(x,y) = Y(fx,y)$ and $f^*\colon Y \mathrel{-\!\!\circ\!\!\to} X$, $f^*(y,x) = Y(y,fx)$. We further observe that for any element $x\colon 1 \to X$ (1 is the one-element $\mathcal{Q}$-category with structure $1(\star,\star) = \mathbf{1}$), the module $x^*\colon X \mathrel{-\!\!\circ\!\!\to} 1$ is in fact the same as the $\mathcal{Q}$-functor $\mathsf{y}_X x = X(-,x) \in \widehat{X}$. Dually, the $\mathcal{Q}$-module $x_*\colon 1 \mathrel{-\!\!\circ\!\!\to} X$ corresponds to the $\mathcal{Q}$-functor $\lambda_X x := X(x,-)$. The order relation between $\mathcal{Q}$-modules can be lifted to $\mathcal{Q}$-functors by putting

$$f \leq g \text{ whenever } f^* \leq g^*,$$

for $\mathcal{Q}$-functors $f, g : X \to Y$, and this makes $\mathcal{Q}$-**Cat** an ordered category. Also note that $1_X^* = X = (1_X)_*$, for every $\mathcal{Q}$-category X. For $x, y \in X$,

$$x \leqslant_X y \iff x^* \leqslant y^* \iff \mathbf{1} \leqslant X(x,y),$$

which defines a preorder on X. A $\mathcal{Q}$-category X is called *separated* if $\leqslant_X$ is antisymmetric. Clearly, $\mathcal{Q}$-functors are $\leqslant_X$-preserving, and therefore this construction yields a functor $\mathcal{Q}$-**Cat** $\to$ **Ord**. For example a separated $[0,\infty]$-category is a so-called quasi-metric space, where points can possibly be at infinite distance.

In **Ord**, modules of type $X \mathrel{-\!\!\circ\!\!\to} 1$ are precisely (characteristic maps of) lower sets, and modules of type $1 \mathrel{-\!\!\circ\!\!\to} X$ are upper sets of the poset X. Surprisingly, the important notion of Cauchy sequence in a metric space relates to a certain kind of modules, as exposed in [23]. In fact, any Cauchy sequence $(x_n)_{n\in\omega}$ induces a module $\phi\colon 1 \mathrel{-\!\!\circ\!\!\to} X$ via $\phi(x) = \lim_{n\to\infty} X(x_n, x)$, and a module $\psi\colon X \mathrel{-\!\!\circ\!\!\to} 1$ via $\psi(x) = \lim_{n\to\infty} X(x, x_n)$. Observe that $\psi \cdot \phi \leqslant 0$ and $\phi \cdot \psi \geqslant X$ in the pointwise order. Conversely, any pair of modules that satisfies the above equations comes from some Cauchy sequence on X. More generally, we will say that $\mathcal{Q}$-modules $\phi\colon Z \mathrel{-\!\!\circ\!\!\to} X$, $\psi\colon X \mathrel{-\!\!\circ\!\!\to} Z$ are adjoint iff $\phi \cdot \psi \leqslant X$ and $\psi \cdot \phi \geqslant Z$. In this case we say that ϕ is a left adjoint to ψ and ψ is a right adjoint to ϕ.

2.4. ***J*-cocomplete $\mathcal{Q}$-categories.** We recall here briefly the notions of weighted limit and weighted colimit, for further details we refer to [16, 18]. The role model is of course the notion of infimum (respectively supremum) in an ordered set which can be formulated in the language of modules as follows. An upper set in an ordered set is a module $\phi : 1 \mathrel{-\!\!\circ\!\!\to} X$, and one immediately verifies that the lower set of lower bounds is the lifting $\phi -\!\!\bullet 1_X^* : X \mathrel{-\!\!\circ\!\!\to} 1$ of the order relation $\leq = 1_X^*$ on X along ϕ. By definition, $x \in X$ is an infimum of ϕ if the lower set of lower bounds of ϕ is generated by x, which means precisely $x^* = \phi \multimap 1_X^*$. In the sequel it will be useful to consider a more flexible notion and think of an upper set in X as the image of an upper set $\phi\colon 1 \mathrel{-\!\!\circ\!\!\to} I$ in some ordered set I along a monotone map $h : I \to X$, and one easily verifies that a supremum of this lower set is an element $x \in X$ with $x^* = \phi \multimap h^*$.

Hence, for a $\mathcal{Q}$-module $\phi\colon 1 \mathrel{-\!\!\circ\!\!\to} I$, a *$\phi$-weighted limit* of a $\mathcal{Q}$-functor $h\colon I \to X$ is an element $x \in X$ with $x^* = \phi \multimap h^*$. Dually, for a module $\psi\colon I \mathrel{-\!\!\circ\!\!\to} 1$, a *$\psi$-weighted colimit* of a $\mathcal{Q}$-functor $h\colon I \to X$ is an element $x \in X$ with $x_* = h_* \multimapinv \psi$. A $\mathcal{Q}$-category X is called *complete* if X admits all weighted limits, and *cocomplete* if X admits all weighted colimits. For instance, $\mathcal{Q}$ is both complete and cocomplete where the limit of h and ϕ is given by $\bigwedge_{i\in I} \mathcal{Q}(\phi(i), h(i))$ and the colimit of h and ψ by $\bigvee_{i\in I} \psi(i) \otimes h(i)$. This argument extends pointwise to $\widehat{X}$, and we also note that a $\mathcal{Q}$-category X is complete if and only if X is cocomplete.

One says that a $\mathcal{Q}$-functor $f : X \to Y$ preserves the ϕ-weighted limit x of $h\colon I \to X$ if $f(x)$ is a ϕ-weighted limit of $fh : I \to Y$, likewise, $f : X \to Y$ preserves the ψ-weighted colimit x of $h\colon I \to X$ if $f(x)$ is a ψ-weighted colimit of $fh : I \to Y$. Then $f : X \to Y$ is called *continuous* if f preserves all existing weighted limits in X, and f is called *cocontinuous* if f preserves all existing weighted colimits in X.

In domain theory one is typically interested not in all but just the directed suprema. Similarly, in the sequel we will consider special kinds of colimits, hence we suppose that there is given a collection J of modules of type $X \mathrel{-\!\!\circ\!\!\to} 1$, called hereafter *$J$-ideals*. The set of those modules in J with domain X we denote as JX. Then we define X to be *J-cocomplete* if X admits all ψ-weighted colimits with ψ in J, and a $\mathcal{Q}$-functor $f : X \to Y$ is called *J-cocontinuous* if f preserves all existing J-weighted colimits in X. We will also assume that our class J of modules is *saturated*, which amounts to saying that JX contains all modules $x^* : X \mathrel{-\!\!\circ\!\!\to} 1$ and is closed in $\widehat{X}$ under J-weighted colimits. In this case, X is J-cocomplete if and only if X admits all ψ-weighted colimits with $\psi : X \mathrel{-\!\!\circ\!\!\to} 1$ in JX, which in turn is equivalent to $\mathsf{y}_X\colon X \to JX$ having a left adjoint in $\mathcal{Q}$-$\mathbf{Cat}$. That is, there must exist a $\mathcal{Q}$-functor $\mathsf{S}_X\colon JX \to X$ such that for all $\phi \in JX$ and all $x \in X$:

$$X(\mathsf{S}_X\phi, x) = \widehat{X}(\phi, \mathsf{y}_X x). \tag{2.1}$$

The element $\mathsf{S}_X\phi \in X$ is called the *supremum* of ϕ. The category of J-cocomplete $\mathcal{Q}$-categories and J-cocontinuous $\mathcal{Q}$-functors will be denoted by J-$\mathbf{Cocts}$.

If $JX = \widehat{X}$ and $\Psi : \widehat{X} \mathrel{-\!\!\circ\!\!\to} 1$, then $\mathsf{S}_X(\Psi)(x) = \bigvee_{\psi\in\widehat{X}} \Psi(\psi) \otimes \psi(x) = \bigvee_{\psi\in\widehat{X}} \Psi(\psi) \otimes [\mathsf{y}_X(x), \psi]$, hence $\mathsf{S}_X(\Psi) = \Psi \cdot (\mathsf{y}_X)_*$. Since JX is closed in $\widehat{X}$ under J-colimits, the same formula describes J-suprema in JX. For example, if $\mathcal{Q} = \mathbf{2}$, then $\widehat{X}$ is a poset of lower subsets of the poset X ordered by inclusion, ψ is a lower set of lower sets of X, and the supremum of ψ is nothing else but $\bigcup \psi$.

A $\mathcal{Q}$-functor $f\colon X \to Y$ between J-cocomplete $\mathcal{Q}$-categories is J-cocontinuous if and only if $f(\mathsf{S}_X\phi) = \mathsf{S}_Y(Jf(\phi))$, for all $\phi \in JX$. Here we make use of the fact that J defines a functor $J : \mathcal{Q}$-$\mathbf{Cat} \to J$-$\mathbf{Cocts}$ which sends a $\mathcal{Q}$-category X to JX, and a $\mathcal{Q}$-functor $f : X \to Y$ to $Jf : JX \to JY$, $\psi \mapsto \psi \cdot f^*$. We use the occasion to remark that $J : \mathcal{Q}$-$\mathbf{Cat} \to J$-$\mathbf{Cocts}$ is left adjoint to the inclusion functor J-$\mathbf{Cocts} \to \mathcal{Q}$-$\mathbf{Cat}$. Even better, J-$\mathbf{Cocts} \to \mathcal{Q}$-$\mathbf{Cat}$ is monadic (see [17]) which we need here only to conclude that J-**Cocts** is complete and limits in J-**Cocts** are calculated as in $\mathcal{Q}$-**Cat**.

There is a well-known general procedure to specify a saturated class J of modules which we describe now.

Example 2.6. Fix a collection Φ of modules $\phi : 1 \to\!\!\!\!\circ\ I$, and define J as the class of all those modules $\psi : X \to\!\!\!\!\circ\ 1$ where the $\mathcal{Q}$-functors

$$\psi \cdot - : \mathcal{Q}^X \to \mathcal{Q},\ \alpha \mapsto \psi \cdot \alpha = \bigvee_{x \in X} \alpha(x) \otimes \psi(x).$$

preserve Φ-weighted limits. Here we identify a $\mathcal{Q}$-functor $\alpha : X \to \mathcal{Q}$ with a module $\alpha : 1 \to\!\!\!\!\circ\ X$. Explicitly, we require that, for any $\phi : 1 \to\!\!\!\!\circ\ I$ in Φ and any $\mathcal{Q}$-functor $\alpha_- : I \to \mathcal{Q}^X$,

$$\bigwedge_{i \in I} \mathcal{Q}(\phi(i), \bigvee_{x \in X} \alpha_i(x) \otimes \psi(x)) = \bigvee_{x \in X} \left(\bigwedge_{i \in I} \mathcal{Q}(\phi(i), \alpha_i(x)) \right) \otimes \psi(x).$$

Note that $\mathcal{Q}$-functoriality of $\psi \cdot -$ implies already that the left hand side is larger or equal to the right hand side.

Cocompleteness relative to J allows for a unified presentation of seemingly unrelated notions of order- and metric completeness:

Example 2.7. For any $\mathcal{Q}$, there is a largest and a smallest choice of J: let either J consist of all modules of type $X \to\!\!\!\!\circ\ 1$, or only of representable modules $x^* : X \to\!\!\!\!\circ\ 1$ where $x \in X$. In the first case a $\mathcal{Q}$-category X is J-cocomplete if and only if it is cocomplete, and in the second case every $\mathcal{Q}$-category is J-cocomplete. We also point out that the situation for $\mathcal{Q}$-categories differs here from the one for ordinary categories where the existence of all colimits does not guarantee the existence of a left adjoint to the Yoneda embedding. Categories admitting such a left adjoint are called *total* (see [29]).

Example 2.8. For $\mathcal{Q} = \mathbf{2}$, we consider all modules of type $X \to\!\!\!\!\circ\ 1$ corresponding to order-ideals in X (i.e. directed and lower subsets of X), and write $J = \mathbf{Idl}$. Then X is **Idl**-cocomplete if and only if X is directed-complete.

Example 2.9. A sequence $(x_n)_{n \in \omega}$ in a $[0,\infty]$-category X is forward Cauchy [4] if

$$\forall \varepsilon > 0\ \exists N\ \forall n \geqslant m \geqslant N\ X(x_n, x_m) \leqslant \varepsilon.$$

For a given forward Cauchy sequence $(x_n)_{n \in \omega}$ consider a map $\phi\colon X^{\mathrm{op}} \to [0,\infty]$ given by $\phi(x) = \sup_n \inf_{m \geq n} X(x, x_m)$ and then consider the class $J = \mathbf{FC}$ of all modules of type $X \to\!\!\!\!\circ\ 1$ corresponding to above defined maps of type $X^{\mathrm{op}} \to [0,\infty]$. As it happens, **FC**-ideals correspond to equivalence classes of forward Cauchy sequences on X. Hence, X is **FC**-cocomplete if and only if each forward Cauchy sequence on X converges if and only if X is sequentially Yoneda complete.

Example 2.10. For any $\mathcal{Q}$ we can choose J to consist of all right adjoint modules (i.e. modules that have left adjoints). Recall from [23] that, for $\mathcal{Q} = [0,\infty]$, a right adjoint module $X \to\!\!\!\!\circ\ 1$ corresponds to an equivalence class of Cauchy sequences on X. A generalised metric space X is J-cocomplete if and only if each Cauchy sequence on X converges.

Example 2.11. For a completely distributive quantale $\mathcal{Q}$ with totally below relation $\prec$ and any $\mathcal{Q}$-category X, a module $\psi\colon X \to\!\!\!\!\circ\ 1$ is a **FSW**-ideal (the acronym "**FSW**" refers to the authors of [11]) if: (a) $\bigvee_{z \in X} \psi z = \mathbf{1}$, and (b) for all $e_1, e_2, d \prec \mathbf{1}$, for all $x_1, x_2 \in X$, whenever $e_1 \prec \psi x_1$

and $e_2 \prec \psi x_2$, then there exists $z \in X$ such that $d \prec \psi z$, $e_1 \prec X(x_1, z)$ and $e_2 \prec X(x_2, z)$. Now for $\mathcal{Q} = [0, \infty]$ **FSW**-ideals on X are in a bijective correspondence with equivalence classes of forward Cauchy nets on X [11]; for $\mathcal{Q} = \mathbf{2}$, **FSW**-ideals are characteristic maps of order-ideals on X. Therefore this example unifies Examples 2.8, 2.9.

Example 2.12. For any quantale $\mathcal{Q}$, a module $\psi\colon X \relbar\joinrel\mapsto 1$ is called *flat* if the map $(\psi \cdot -)$ taking modules of type $1 \relbar\joinrel\mapsto X$ to $\mathcal{Q}$ preserves finite meets. For $\mathcal{Q} = \mathbf{2}$, one verifies that $\psi : X \relbar\joinrel\mapsto 1$ is flat if and only if $\psi : X^{\mathrm{op}} \to \mathbf{2}$ is the characteristic map of a directed down-set. For $\mathcal{Q} = [0, \infty]$ with $\otimes = +$, Theorem 7.15 of [31] states that flat modules are the same as **FSW**-ideals, therefore this example unifies Examples 2.8, 2.9 as well. However, as we will show in Subsection 4.3, flat modules and **FSW**-ideals are in general different.

Example 2.13. For any $\mathcal{Q}$, put JX to be the set of all modules $\psi : X \relbar\joinrel\mapsto 1$ of the form $\psi = u \cdot x^*$ where $x \in X$ and $u \in Q$. Here we think of $u \in Q$ as a module $1 \relbar\joinrel\mapsto 1$. Spelled out, for $y \in X$ one has $\psi(y) = X(y, x) \otimes u$. Note that $\psi(y) = \bot$ whenever $u = \bot$, independently of $x \in X$. A $\mathcal{Q}$-category X is J-cocomplete if it admits "tensoring" with elements of $\mathcal{Q}$ in the following sense: for any $x \in X$ and $u \in Q$, there exists a (necessarily unique up to equivalence) element $z \in X$ with

$$X(z, y) = \mathcal{Q}(u, X(x, y))$$

for all $y \in X$, and one denotes z as $u \otimes x$.

2.5. **J-continuous J-cocomplete $\mathcal{Q}$-categories.** J-continuity for $\mathcal{Q}$-categories, studied extensively in [14], allows for a unified treatment of many structures that play a major role in theoretical computer science, e.g. continuous domains, complete metric spaces, or completely distributive complete lattices.

Definition 2.14. A J-cocomplete $\mathcal{Q}$-category X is *J-continuous* if the supremum $\mathsf{S}_X\colon JX \to X$ has a left adjoint.

Note that any $\mathcal{Q}$-functor of type $X \to JX$ corresponds to a certain module $X \relbar\joinrel\mapsto X$ belonging to J. Hence, X is J-continuous if and only if there exists a module $\Downarrow_X\colon X \relbar\joinrel\mapsto X$ in J with $\ulcorner \Downarrow_X \urcorner \dashv \mathsf{S}_X$. It is not difficult to see that $\mathsf{S}_X^* \cdot \Downarrow_X \leqslant \mathsf{y}_{X*}$, and $\Downarrow_X$ is the largest module that satisfies this inequality; hence we have identified $\Downarrow_X\colon X \relbar\joinrel\mapsto X$ as the lifting $\Downarrow_X = \mathsf{S}_X^* \multimap \mathsf{y}_{X*}$. In fact, the module $\Downarrow_X := \mathsf{S}_X^* \multimap \mathsf{y}_{X*}$ exists for any J-cocomplete $\mathcal{Q}$-category, and we refer to it as the *way-below* module. It is worth noting that JX is J-continuous for every $\mathcal{Q}$-category X. In this case, the way-below module is given by

$$\Downarrow(\psi, \psi') = \bigvee_{x \in X} \psi'(x) \otimes [\psi, x^*]. \tag{2.2}$$

In the simplest case, $\mathcal{Q} = \mathbf{2}$ and $J = \mathbf{Idl}$, the module $\Downarrow_X$ is indeed the (characteristic map of the) way-below relation on X. In the case of metric spaces, as a consequence of symmetry, $\Downarrow_X\colon X \relbar\joinrel\mapsto X$ is the same as the structure $X\colon X \relbar\joinrel\mapsto X$.

We call a module $v\colon X \relbar\joinrel\mapsto X$ *auxiliary*, if $v \leqslant X$; *interpolative*, if $v \leqslant v \cdot v$; *approximating*, if $v \in J$ and $X \multimapinv v = X$; *J-cocontinuous*, if $\mathsf{S}_X^* \cdot v = \mathsf{y}_{X*} \cdot v$. In a J-continuous J-cocomplete $\mathcal{Q}$-category, the way-below module is auxiliary, interpolative, approximating and J-cocontinuous. In fact, we show [14] that a J-cocomplete $\mathcal{Q}$-category is J-continuous iff the way-below module is approximating.

Consider some examples: **FSW**-continuous **FSW**-cocomplete **2**-categories are precisely continuous domains; cocontinuous cocomplete **2**-categories are completely distributive complete lattices (there the way-below module becomes the 'totally-below' relation associated with complete distributivity of the underlying lattice); $[0,\infty]$ considered with the generalised metric structure $[0,\infty](x,y) = \max\{y-x,0\}$ is an **FSW**-continuous **FSW**-complete $[0,\infty]$-category; complete metric spaces are **FSW**-continuous **FSW**-cocomplete $[0,\infty]$-categories.

2.6. **Open modules.** Recall that J-**Cocts**(X,Y) denotes the set of all J-cocontinuous $\mathcal{Q}$-functors from X to Y, and we view J-**Cocts**$(X,\mathcal{Q})$ as a sub-$\mathcal{Q}$-category of $\mathcal{Q}^X$.

Lemma 2.15. *J-**Cocts**$(X,\mathcal{Q})$ is closed under arbitrary suprema in $\mathcal{Q}^X$. Hence, J-**Cocts**$(X,\mathcal{Q})$ is cocomplete.*

Proof. Just observe that $\bigvee\colon \mathcal{Q}^I \to \mathcal{Q}$ is a $\mathcal{Q}$-functor left adjoint to the diagonal $\Delta\colon \mathcal{Q} \to \mathcal{Q}^I$, for any set I; and $u \otimes -\colon \mathcal{Q} \to \mathcal{Q}$ is a $\mathcal{Q}$-functor left adjoint to $\mathcal{Q}(u,-)\colon \mathcal{Q} \to \mathcal{Q}$. □

From the lemma above we deduce that the inclusion functor J-**Cocts**$(X,\mathcal{Q}) \hookrightarrow \mathcal{Q}^X$ has a right adjoint $v\colon \mathcal{Q}^X \to J$-**Cocts**$(X,\mathcal{Q})$.

If X is J-cocomplete and J-continuous, this right adjoint has a simple description. In fact, since $\Downarrow_X \dashv \mathsf{S}_X$ and $\mathsf{S}_X \dashv \mathsf{y}_X$, the map $\mathcal{Q}^X \to J$-**Cocts**$(X,\mathcal{Q})$, $f \mapsto f_L \cdot \Downarrow_X$ (where f_L is left Kan extension of f) is right adjoint to J-**Cocts**$(X,\mathcal{Q}) \hookrightarrow \mathcal{Q}^X$ in **Ord**, hence it underlies v. Hence in this case we can write v as the corestriction of the composite of left adjoints

$$\mathcal{Q}^X \longrightarrow J\text{-}\mathbf{Cocts}(JX,\mathcal{Q}) \hookrightarrow \mathcal{Q}^{JX} \xrightarrow{-\cdot\Downarrow_X} \mathcal{Q}^X$$

to J-**Cocts**$(X,\mathcal{Q})$, hence v is itself left-adjoint.

Lemma 2.16. *If X is J-cocomplete and J-continuous, then J-**Cocts**$(X,\mathcal{Q})$ is continuous with respect to the class of all $\mathcal{Q}$-modules (totally continuous for short).*

Proof. $\mathcal{Q}^X$ is totally continuous, and J-**Cocts**$(X,\mathcal{Q})$ inherits this property since $v\colon \mathcal{Q}^X \to J$-**Cocts**$(X,\mathcal{Q})$ is a left and a right adjoint. □

We put now $FX := J$-**Cocts**$(X,\mathcal{Q}) \cap J(X^{\mathrm{op}})$ and call $\alpha \in FX$ an *open module*. More precisely, FX is defined via the pullback in J-**Cocts** of two inclusions: J-**Cocts**$(X,\mathcal{Q}) \hookrightarrow \mathcal{Q}^X$, $J(X^{\mathrm{op}}) \hookrightarrow \mathcal{Q}^X$, which tells us that:

- FX is J-cocomplete,
- both inclusion maps $FX \hookrightarrow J(X^{\mathrm{op}})$ and $FX \hookrightarrow J$-**Cocts**$(X,\mathcal{Q})$ preserve J-suprema.

Definition 2.17. We say that a J-continuous $\mathcal{Q}$-category X is *open module determined* if for all $x,y \in X$:

$$\Downarrow_X(x,y) = \bigvee_{\alpha\in FX} (\alpha(y) \otimes [\alpha, \lambda_X(x)]). \tag{2.3}$$

Note that, for all $\alpha \in FX$ and $x,y \in X$,

$$\alpha(y) \otimes [\alpha, \lambda_X(x)] = \bigvee_{z\in X} (\alpha(z) \otimes \Downarrow_X(z,y) \otimes [\alpha, X(x,-)]) \leqslant \bigvee_{z\in X} X(x,z) \otimes \Downarrow_X(z,y) = \Downarrow_X(x,y),$$

hence (2.3) is equivalent to

$$\Downarrow_X(x,y) \leqslant \bigvee_{\alpha\in FX} (\alpha(y) \otimes [\alpha, \lambda_X(x)]).$$

Furthermore, (2.3) is equivalent to

$$\Downarrow_X(x,y) = \bigvee_{\alpha\in FX} (\alpha(y) \otimes [\alpha, \Downarrow_X(x,-)])$$

since $\Downarrow_X(x,-)\leqslant \lambda_X(x)$ and

$$\begin{aligned}\Downarrow_X(x,y) &= \bigvee_{z\in X}\Downarrow_X(x,z)\otimes\Downarrow_X(z,y)\\ &= \bigvee_{z\in X}\Downarrow_X(x,z)\otimes\bigvee_{\alpha\in FX}(\alpha(y)\otimes[\alpha,\lambda_X(z)])\\ &= \bigvee_{\alpha\in FX}\alpha(y)\otimes\bigvee_{z\in X}(\Downarrow_X(x,z)\otimes[\alpha,\lambda_X(z)])\\ &\leqslant \bigvee_{\alpha\in FX}\alpha(y)\otimes[\alpha,\bigvee_{z\in X}\Downarrow_X(x,z)\otimes X(z,-)]\\ &= \bigvee_{\alpha\in FX}(\alpha(y)\otimes[\alpha,\Downarrow_X(x,-)]).\end{aligned}$$

3. The duality

In this section we assume that a class Φ of limit weights $\phi : 1 \mathrel{-\!\!\circ\!\!\rightarrow} I$ is given, and we consider the corresponding class J of modules as described in Example 2.6. Furthermore, let X be a J-cocomplete, J-continuous and open module determined $\mathcal{Q}$-category.

Each $x\in X$ defines:

$$\begin{aligned}\mathrm{ev}_x\colon FX&\to\mathcal{Q}\\ \alpha&\mapsto\alpha(x).\end{aligned}$$

Lemma 3.1. *For any $x\in X$, the map ev_x is an open module on FX.*

Proof. Certainly, ev_x is J-continuous, since it is the restriction of

$$-\cdot x_*\colon J(X^{\mathrm{op}})\to\mathcal{Q}\qquad\qquad(\text{here } x\in X^{\mathrm{op}} \text{ and therefore } x_*\colon 1\mathrel{-\!\!\circ\!\!\rightarrow} X^{\mathrm{op}})$$

to FX. We show now that $\mathrm{ev}_x\in J(FX^{\mathrm{op}})$, that is,

$$C_x:=\mathrm{ev}_x\cdot -\colon \mathcal{Q}\text{-}\mathbf{Mod}(FX,1)\to\mathcal{Q},\ \Psi\mapsto\bigvee_{\alpha\in FX}\Psi(\alpha)\otimes\alpha(x)$$

preserves Φ-weighted limits. Note that $\mathcal{Q}\text{-}\mathbf{Mod}(FX,1)\cong\mathcal{Q}\text{-}\mathbf{Mod}(1,FX^{\mathrm{op}})$. Furthermore, since $\alpha\in FX$ is J-cocontinuous, $C_x=\bigvee_{y\in X}C_y\otimes\Downarrow_X(y,x)$. Let $\phi:1\mathrel{-\!\!\circ\!\!\rightarrow} I$ be in Φ and $\Psi_-\colon I\to\mathcal{Q}\text{-}\mathbf{Mod}(FX,1)$, $i\mapsto\Psi_i$ be a $\mathcal{Q}$-functor. Then

$$\begin{aligned}\bigwedge_{i\in I}\mathcal{Q}(\phi(i),C_x(\Psi_i)) &= \bigwedge_{i\in I}\mathcal{Q}(\phi(i),\bigvee_{y\in X}C_y(\Psi_i)\otimes\Downarrow_X(y,x))\\ &= \bigvee_{y\in X}\left(\bigwedge_{i\in I}\mathcal{Q}(\phi(i),C_y(\Psi_i))\right)\otimes\Downarrow_X(y,x) \qquad\qquad (\Downarrow(-,x)\text{ is in } J)\\ &\leqslant \bigvee_{\alpha\in FX}\alpha(x)\otimes\bigvee_{y\in X}\bigwedge_{i\in I}\mathcal{Q}(\phi(i),C_y(\Psi_i)\otimes[\alpha,\lambda_X y])\\ &\leqslant \bigvee_{\alpha\in FX}\alpha(x)\otimes\bigwedge_{i\in I}\mathcal{Q}(\phi(i),\Psi_i(\alpha))\end{aligned}$$

since

$$C_y(\Psi_i)\otimes[\alpha,\lambda_X y]=\bigvee_{\beta\in FX}\Psi_i(\beta)\otimes[\alpha,\lambda_X y]\otimes[\lambda_X y,\beta]\leqslant\bigvee_{\beta\in FX}\Psi_i(\beta)\otimes[\alpha,\beta]=\Psi_i(\alpha).\qquad\square$$

We further obtain a map $\eta_X\colon X \to FFX$ given by:

$$x \mapsto \mathrm{ev}_x. \tag{3.1}$$

This is indeed a $\mathcal{Q}$-functor, since for any $y, z \in X$ we have:

$$[\eta_X(y), \eta_X(z)] = \bigwedge_{\alpha \in FX} \mathcal{Q}(\alpha(y), \alpha(z)) \geqslant X(y, z).$$

Lemma 3.2. *FX is J-continuous with the way-below module $\Downarrow_{FX}\colon FX \,{\circ\!\!\to}\, FX$ given by:*

$$\Downarrow_{FX}(\beta, \alpha) = \bigvee_{x \in X} (\alpha(x) \otimes [\beta, \lambda_X(x)]). \tag{3.2}$$

Proof. Note that (3.2) states that the way-below module on FX is the restriction of the way-below module on $J(X^{\mathrm{op}})$ (see (2.2)). First we wish to show that

$$\Downarrow_{FX}(-, \alpha) := \bigvee_{x \in X} (\alpha(x) \otimes [-, \lambda_X(x)])$$

is a J-module of type $FX \,{\circ\!\!\to}\, 1$, for every $\alpha \in FX$. To this end, we consider a diagram

$$1 \xrightarrow{\phi}\!\!\!\!\!\circ\; A \xrightarrow{h} \mathcal{Q}^{FX}$$

where ϕ belongs to Φ. We calculate:

$$\begin{aligned}
&\bigwedge_{a \in A} \mathcal{Q}(\phi(a), \bigvee_{\beta \in FX} (\Downarrow_{FX}(\beta, \alpha) \otimes h(a, \beta)))\\
&= \bigwedge_{a \in A} \mathcal{Q}(\phi(a), \bigvee_{x \in X} (\alpha(x) \otimes (\bigvee_{\beta \in FX} ([\beta, \lambda_X(x)] \otimes h(a, \beta)))))\\
&\{\text{put } k(a, x) := \bigvee_{\beta \in FX} ([\beta, \lambda_X(x)] \otimes h(a, \beta)) \text{ where } k\colon A \to \mathcal{Q}^{X^{\mathrm{op}}}\}\\
&= \bigvee_{x \in X} (\alpha(x) \otimes \bigwedge_{a \in A} (\mathcal{Q}(\phi(a), k(a, x))))\\
&= \bigvee_{x, y \in X} ((\alpha(y) \otimes \Downarrow_X(y, x)) \otimes \bigwedge_{a \in A} (\mathcal{Q}(\phi(a), k(a, x))))\\
&= \bigvee_{\gamma \in FX} \bigvee_{x, y \in X} ((\gamma(x) \otimes \alpha(y) \otimes [\gamma, \lambda_X(y)]) \otimes \bigwedge_{a \in A} (\mathcal{Q}(\phi(a), k(a, x))))\\
&= \bigvee_{\gamma \in FX} \bigvee_{y \in X} (\alpha(y) \otimes [\gamma, \lambda_X(y)] \otimes (\bigvee_{x \in X} (\gamma(x) \otimes \bigwedge_{a \in A} (\mathcal{Q}(\phi(a), k(a, x))))))\\
&= \bigvee_{\gamma \in FX} (\Downarrow_{FX}(\gamma, \alpha) \otimes \bigwedge_{a \in A} (\mathcal{Q}(\phi(a), \bigvee_{x \in X} (\gamma(x) \otimes k(a, x)))))\\
&= \bigvee_{\gamma \in FX} (\Downarrow_{FX}(\gamma, \alpha) \otimes \bigwedge_{a \in A} (\mathcal{Q}(\phi(a), \bigvee_{\beta \in FX} \bigvee_{x \in X} (\gamma(x) \otimes [\beta, \lambda_X(x)] \otimes h(a, \beta)))))\\
&= \bigvee_{\gamma \in FX} (\Downarrow_{FX}(\gamma, \alpha) \otimes \bigwedge_{a \in A} (\mathcal{Q}(\phi(a), \bigvee_{\beta \in FX} ([\beta, \gamma] \otimes h(a, \beta)))))\\
&\leqslant \bigvee_{\gamma \in FX} (\Downarrow_{FX}(\gamma, \alpha) \otimes \bigwedge_{a \in A} (\mathcal{Q}(\phi(a), h(a, \beta)))),
\end{aligned}$$

as required (recall that the other inequality we get for free). Furthermore, we calculate:

$$\begin{aligned}
\mathsf{S}_{FX}(\Downarrow_{FX}(-,\alpha))(x) &= \bigvee_{\beta\in FX}(\Downarrow_{FX}(\beta,\alpha)\otimes\beta(x))\\
&= \bigvee_{\beta\in FX}\bigvee_{y\in X}(\alpha(y)\otimes[\beta,\lambda_X(y)]\otimes\beta(x))\\
&= \bigvee_{y\in X}(\alpha(y)\otimes\bigvee_{\beta\in FX}([\beta,\lambda_X(y)]\otimes\beta(x)))\\
&= \bigvee_{y\in X}(\alpha(y)\otimes\Downarrow_X(y,x))\\
&= \alpha(x),
\end{aligned}$$

hence $\mathsf{S}_{FX}(\Downarrow_{FX}(-,\alpha))=\alpha$. Finally, to conclude that $\ulcorner\Downarrow_{FX}^{\urcorner}\dashv \mathsf{y}_{FX}$, let $\psi: FX \mathrel{-\!\!\circ\!\!\to} 1$ in J. Let i denote the inclusion $\mathcal{Q}$-functor $FX\hookrightarrow J(X^{\mathrm{op}})$ and $\Downarrow_{J(X^{\mathrm{op}})}$ the way-below module on $J(X^{\mathrm{op}})$. We observed already that $\Downarrow_{FX}=i^*\cdot\Downarrow_{J(X^{\mathrm{op}})}\cdot i_*$. Hence,

$$\begin{aligned}
\ulcorner\Downarrow_{FX}^{\urcorner}\cdot\mathsf{S}_{FX}(\psi) = (\mathsf{S}_{FX}(\psi))^*\cdot\Downarrow_{FX} &= (\mathsf{S}_{FX}(\psi))^*\cdot i^*\cdot\Downarrow_{J(X^{\mathrm{op}})}\cdot i_*\\
&= (\mathsf{S}_{J(X^{\mathrm{op}})}(\psi\cdot i^*))^*\cdot\Downarrow_{J(X^{\mathrm{op}})}\cdot i_*\leqslant\psi\cdot i^*\cdot i_*=\psi. \qquad\square
\end{aligned}$$

Lemma 3.3. *FX is open module determined.*

Proof. For all $\alpha,\beta\in FX$:

$$\begin{aligned}
\Downarrow_{FX}(\beta,\alpha) = \bigvee_{z\in X}(\alpha(z)\otimes[\beta,\lambda_X(z)]) &= \bigvee_{z\in X}(\mathrm{ev}_z(\alpha)\otimes[\lambda_X(z)_*,\beta_*])\\
&= \bigvee_{z\in X}(\mathrm{ev}_z(\alpha)\otimes[\mathrm{ev}_z,\lambda_{FX}(\beta)]) = \bigvee_{\mathcal{A}\in FFX}(\mathcal{A}(\alpha)\otimes[\mathcal{A},\lambda_{FX}(\beta)]) \qquad\square
\end{aligned}$$

By the discussion in Section 2.6 and Lemmata 3.2, 3.3 we obtain:

Theorem 3.4. *If X is a J-continuous, J-cocomplete and open module determined $\mathcal{Q}$-category, then so is FX.*

Our next aim is to show that $\eta_X\colon\ : X\to FFX$ is an isomorphism. To do so, let now $\mathcal{A}\colon FX\to\mathcal{Q}$ be an open module on FX. We define:

$$\psi_{\mathcal{A}}(x):=\bigvee_{\alpha\in FX}(\mathcal{A}(\alpha)\otimes[\alpha,\lambda_X(x)]).$$

Such defined $\psi_{\mathcal{A}}$ is a module $X\mathrel{-\!\!\circ\!\!\to}1$, since it is the composite:

$$X\xrightarrow{\lambda_{X*}}J(X^{\mathrm{op}})^{\mathrm{op}}\xrightarrow{i^*}FX^{\mathrm{op}}\xrightarrow{\mathcal{A}}1.$$

We also need to have:

Lemma 3.5. *For every $\mathcal{A}\in FFX$, we have $\psi_{\mathcal{A}}\in JX$.*

Proof. In order to check that $\psi_{\mathcal{A}}\colon X\mathrel{-\!\!\circ\!\!\to}1$ belongs to JX, we need to check whether $\psi_{\mathcal{A}}\cdot-\colon\mathcal{Q}^X\to\mathcal{Q}$ preserves Φ-weighted limits. Let

$$1\xrightarrow{\phi}A\xrightarrow{h}\mathcal{Q}^X$$

be a limit diagram with ϕ in Φ. Spelled out, we have to show that

$$\bigvee_{x\in X}(\psi_{\mathcal{A}}(x)\otimes\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),h(y,x))))\geqslant\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\bigvee_{x\in X}(\psi_{\mathcal{A}}(x)\otimes h(y,x)))).$$

To this end, we calculate:

$$\begin{aligned}
&\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\bigvee_{x\in X}(\psi_{\mathcal{A}}(x)\otimes h(y,x))))\\
&=\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\bigvee_{x\in X}\bigvee_{\alpha\in FX}(\mathcal{A}(\alpha)\otimes[\alpha,\lambda_X(x)]\otimes h(y,x))))\\
&=\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\bigvee_{\alpha\in FX}(\mathcal{A}(\alpha)\otimes\Downarrow_{FX}(\alpha,h(y)))))\qquad\{\text{since }\mathcal{A}^{\mathrm{op}}\in J\}\\
&=\bigvee_{\alpha\in FX}(\mathcal{A}(\alpha)\otimes\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\Downarrow_{FX}(\alpha,h(y)))))\\
&=\bigvee_{\alpha,\beta\in FX}((\mathcal{A}(\beta)\otimes\Downarrow_{FX}(\beta,\alpha))\otimes\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\Downarrow_{FX}(\alpha,h(y)))))\\
&=\bigvee_{\alpha,\beta\in FX}\bigvee_{x\in X}((\mathcal{A}(\beta)\otimes\alpha(x)\otimes[\beta,\lambda_X(x)])\otimes\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\Downarrow_{FX}(\alpha,h(y)))))\\
&=\bigvee_{x\in X}\bigvee_{\beta\in FX}(\mathcal{A}(\beta)\otimes[\beta,\lambda_X(x)])\otimes\bigvee_{\alpha\in FX}\mathrm{ev}_x(\alpha)\otimes\bigwedge_{y\in A}(\mathcal{Q}(\phi(y),\Downarrow_{FX}(\alpha,h(y))))\\
&\{\mathrm{ev}_x\text{ is a filter}\}\\
&=\bigvee_{x\in X}(\psi_{\mathcal{A}}(x)\otimes\bigwedge_{y\in X}\mathcal{Q}(\phi(y),\bigvee_{\alpha\in FX}(\alpha(x)\otimes\Downarrow_{FX}(\alpha,h(y)))))\\
&\leqslant\bigvee_{x\in X}(\psi_{\mathcal{A}}(x)\otimes\bigwedge_{y\in X}\mathcal{Q}(\phi(y),\alpha(x)\otimes[\alpha,h(y)]))\\
&\leqslant\bigvee_{x\in X}(\psi_{\mathcal{A}}(x)\otimes\bigwedge_{y\in X}\mathcal{Q}(\phi(y),h(y,x))),
\end{aligned}$$

which proves $\psi_{\mathcal{A}}\in JX$. □

Lemma 3.6. *For any $\alpha\in FX$, we have $\mathcal{A}(\alpha)=\alpha(\mathsf{S}_X(\psi_{\mathcal{A}}))$.*

Proof.

$$\begin{aligned}
\alpha(\mathsf{S}_X(\psi_{\mathcal{A}}))&=\mathrm{colim}(\alpha,\psi_{\mathcal{A}})\\
&=\bigvee_{x\in X}(\alpha(x)\otimes\psi_{\mathcal{A}}(x))\\
&=\bigvee_{x\in X}(\alpha(x)\otimes\bigvee_{\beta\in FX}(\mathcal{A}(\beta)\otimes[\beta,\lambda_X(x))))\\
&=\bigvee_{\beta\in FX}(\mathcal{A}(\beta)\otimes\bigvee_{x\in X}(\alpha(x)\otimes[\beta,\lambda_X(x)]))\\
&=\bigvee_{\beta\in FX}(\mathcal{A}(\beta)\otimes\Downarrow_{FX}(\beta,\alpha))\\
&=\mathrm{colim}(\mathcal{A},\Downarrow_{FX}(-,\alpha))\\
&=\mathcal{A}(\mathsf{S}_{FX}(\Downarrow_{FX}(-,\alpha)))\\
&=\mathcal{A}(\alpha).
\end{aligned}$$

□

Definition 3.7. We say that a $\mathcal{Q}$-functor $f\colon X\to Y$ between $\mathcal{Q}$-categories *reflects open modules* if $\alpha\cdot f\in FX$ for every $\alpha\in FY$. Let $(J,\mathcal{Q})$-**Dom** be the category of J-cocomplete, J-continuous and open module determined $\mathcal{Q}$-categories together with open module reflecting maps.

Lemma 3.8. *The pair of operations*

$$\begin{array}{rcl} X & \mapsto & FX \\ f\colon X \to Y & \mapsto & - \cdot f\colon FY \to FX \end{array}$$

defines a contravariant functor, i.e. $F\colon (J, \mathcal{Q})\text{-}\mathbf{Dom}^{\mathrm{op}} \to (J, \mathcal{Q})\text{-}\mathbf{Dom}$.

Proof. Functoriality is trivial; we only need to show that $F(f)$ reflect open modules. Let $\mathcal{A} \in FFX$. By Lemma 3.6 there exists $x \in X$ such that $\mathcal{A} = \mathrm{ev}_x$, namely $x = \mathsf{S}_X \psi_{\mathcal{A}}$. Then, for any $\alpha \in FY$, we have $(\mathcal{A} \cdot F(f))(\alpha) = \mathcal{A}(\alpha \cdot f) = \alpha(f(x)) = \mathrm{ev}_{f(x)}(\alpha)$. Hence $\mathcal{A} \cdot F(f) = \mathrm{ev}_{f(x)}$, i.e. $\mathcal{A} \cdot F(f) \in FFX$. □

Theorem 3.9 (The Duality Theorem)**.** *The category* $(J, \mathcal{Q})$-**Dom** *is self-dual.*

Proof. The natural isomorphism $\eta\colon 1_{(J,\mathcal{Q})\text{-}\mathbf{Dom}} \to FF$ as defined in (3.1) has the converse $\varepsilon\colon FF \to 1_{(J,\mathcal{Q})\text{-}\mathbf{Dom}}$ given by $\varepsilon_X(\mathcal{A}) = \mathsf{S}_X \psi_{\mathcal{A}}$ for every $\mathcal{A} \in FFX$. □

4. Examples of the duality

4.1. **Lawson duality.** The case $\mathcal{Q} = \mathbf{2}$ and $J = \mathbf{FSW}$, perhaps the simplest possible, served us as a proof guide throughout the paper. In fact, most of the crucial proof ideas (e.g. Lemma 3.6: any open module on open modules $\mathcal{A}$ is of the form $\mathrm{ev}_{\mathsf{S}_X \psi_{\mathcal{A}}}$ for some J-ideal $\psi_{\mathcal{A}}$) come from an analysis of this simple case. Observe that **FSW**-continuous, **FSW**-cocomplete **2**-categories are continuous dcpos (domains). Furthermore, open modules are nothing else but (the characteristic maps of) Scott-open filters on domains. Recall that in this case any FX is open module determined: the equality (2.3) reduces to

$$\forall x, y \in X \ (x \ll y \ \Rightarrow \ \exists \alpha \in FX \ (y \in \alpha \subseteq \twoheaduparrow x)),$$

and we define such $\alpha \in FX$ by $\alpha := \bigcup_{n\in\omega} \twoheaduparrow x_n$, where the descending chain $(x_n)_{n\in\omega}$ has been obtained by a repeated use of interpolation (see Prop. 3.3 of [13]):

$$x \ll \ldots \ll x_n \ll x_{n-1} \ll \ldots \ll x_2 \ll x_1 \ll x_0 = y.$$

Consequently, the category $(\mathbf{FSW}, \mathbf{2})$–**Dom** is the category of domains with open filter reflecting maps; our Theorem 3.9 reduces to Theorem IV-2.12 of [13] establishing the Lawson duality for domains. It is worth mentioning that the Lawson duality (originally proved in [22]) finds its applications in the theory of locally compact spaces; in particular, the lattice of opens of a locally compact sober space X is Lawson dual to the lattice of compact saturated subsets of X (cf. Hofmann-Mislove theorem).

4.2. **A metric duality.** In the case $\mathcal{Q} = [0, \infty]$ with $\otimes = +$ and J being the class of **FSW**-ideals (or, equivalently, flat modules), our duality works in a certain subcategory of **Met**: its **FSW**-cocomplete objects are known in the literature as Yoneda-complete gmses [4]. The **FSW**-cocomplete and **FSW**-continuous ones form a class not previously discussed in the literature, except in the forthcoming paper [21], where they are shown to be precisely the spaces having continuous and directed-complete formal ball models [8, 2, 25] (this implies, in particular, that their topology and metric structure can be respectively characterized as a subspace Scott topology and a partial metric on a domain).

A proof that objects of $(\mathbf{FSW}, [0, \infty])$–**Dom** are open filter determined can be found in [3]; below we present a sketch of the proof.

We abbreviate $\Downarrow_X$ to $\Downarrow$ and customarily use $+$ instead of $\otimes$, inf instead of $\bigvee$, etc. In order to show (2.3) it is enough to find a family of open filters $(\alpha_{e,b})_{e,b>0}$, such that $e > \Downarrow(x,y)$ implies

$$e + b \geqslant \alpha_{e,b}(y) + [\alpha_{e,b}, \Downarrow(x,-)] \geqslant \inf_{\alpha\in FX}(\alpha(y) + [\alpha, \Downarrow(x,-)]),$$

which, by complete distributivity of $([0,\infty],\geqslant)$, allows us to draw the desired conclusion. Take an arbitrary $e > \Downarrow(x,y)$ and $b > 0$, and choose a chain $(e_n)_{n\in\omega}$ in $([0,\infty],\geqslant)$ such that:

$$\begin{aligned} &b > e_0 + e_0,\\ &e_0 > e_1 > e_2 > \ldots > e_n > \ldots > 0,\\ &e_n \geqslant e_{n+1} + e_{n+2} + \ldots,\\ &\inf_{n\in\omega} e_n = 0. \end{aligned} \tag{4.1}$$

Now, by interpolation, we can find a sequence $(x_n)_{n\in\omega}$ such that:

$$\begin{aligned} &e > \Downarrow(x,x_0) + \Downarrow(x_0,y) && \text{and } e_0 > \Downarrow(x_0,y),\\ &e > \Downarrow(x,x_1) + \Downarrow(x_1,x_0) + \Downarrow(x_0,y), && \text{and } e_1 > \Downarrow(x_1,x_0),\\ &e > \Downarrow(x,x_2) + \Downarrow(x_2,x_1) + \Downarrow(x_1,x_0) + \Downarrow(x_0,y) && \text{and } e_2 > \Downarrow(x_2,x_1),\\ &\ldots\\ &e > \Downarrow(x,x_n) + \Downarrow(x_n,x_{n-1}) + \cdots + \Downarrow(x_1,x_0) + \Downarrow(x_0,y) && \text{and } e_n > \Downarrow(x_n,x_{n-1}),\\ &\ldots \end{aligned}$$

Define $\alpha_{e,b}\colon X \to [0,\infty]$ as $\alpha_{e,b}(z) := \inf_{n\in\omega}\sup_{k\geq n} X(x_k,z)$; this map is an open module on X. In order to conclude (2.3), it is now enough to verify that

$$e + b \geqslant \alpha_{e,b}(y) + [\alpha_{e,b}, \Downarrow(x,-)]. \tag{4.2}$$

However

$$\begin{aligned} \alpha_{e,b}(y) &= \inf_{n\in\omega}\sup_{k\geq n} X(x_k,y)\\ &\leqslant \sup_{k\geq 1}\,(X(x_k,x_{k-1}) + \cdots + X(x_1,x_0) + X(x_0,y))\\ &\leqslant \sup_{k\geq 1}\,(\Downarrow(x_k,x_{k-1}) + \cdots + \Downarrow(x_1,x_0) + \Downarrow(x_0,y)) \quad \{\text{by } (4.1)\}\\ &\leqslant e_0 + e_0\\ &< b. \end{aligned}$$

and

$$\begin{aligned} [\alpha_{e,b}, \Downarrow(x,-)] &= \sup_{z\in X}(\Downarrow(x,z) - \alpha_{e,b}(z))\\ &\leqslant \sup_{z\in X}(\Downarrow(x,z) - (\inf_{n\in\omega}\sup_{k\geq n} X(x_k,z)))\\ &\leqslant \sup_{z\in X}(\inf_{n\in\omega}\sup_{k\geq n}(\Downarrow(x,z) - X(x_k,z)))\\ &\leqslant \sup_{n\in\omega}\sup_{k\geq n}\Downarrow(x,x_k)\\ &\leqslant e. \end{aligned}$$

so (4.2), and therefore also (2.3) are now verified.

4.3. **An ultrametric duality.** For the quantale $\mathcal{Q} = [0,\infty]$ with $\otimes = \max$, $\mathcal{Q}$-**Cat** is the category **UMet** of ultrametric spaces and contraction maps. As above, we can choose J to be the class of all flat modules (see Example 2.12), and obtain that the corresponding category $(J,\mathcal{Q})$-**Dom** is self-dual. However, in ultrametric spaces flat modules are not, in general, **FSW**-ideals, as the following example shows. This also answers a question left open in [31].

Example 4.1. Consider the set $\mathbb{N}$ of natural numbers with the distance

$$\mathbb{N}(n,m) = \begin{cases} 0 & \text{if } n = m, \\ \max(n,m) & \text{otherwise.} \end{cases}$$

This distance is a symmetric, separable ultrametric. Take

$$\phi(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Trivially, ϕ preserves the empty meet. Now, observe that the proof of (the equivalence of (1) and (2) of) Proposition 7.9 in [31] holds verbatim for $\otimes = \max$, hence it is enough to show that $(\phi \cdot -)$ preserves meets of modules of the form $\max(\mathbb{N}(-,x),c)$ for some $c \in [0,\infty]$. Suppose $A := \max(\mathbb{N}(-,a),c_1)$ and $B := \max(\mathbb{N}(-,b),c_2)$ for $c_1, c_2 \in [0,\infty]$; we are heading to prove:

$$(*) \qquad \inf_{z\in\mathbb{N}} \max(Az, Bz, \phi z) = \max(\inf_{s\in\mathbb{N}}(\max(As,\phi s)), \inf_{r\in\mathbb{N}}(\max(Br,\phi r))).$$

We have

$$\begin{aligned}
&\inf_{z\in\mathbb{N}} \max(Az, Bz, \phi z) = \inf_{z\in\mathbb{N}} \max(z, a, b, c_1, c_2, \phi z) = \max(a, b, c_1, c_2), \\
&\inf_{s\in\mathbb{N}} \max(As, \phi s) = \inf_{s\in\mathbb{N}} \max(s, a, c_1, \phi s) = \max(a, c_1), \\
&\inf_{r\in\mathbb{N}} \max(Br, \phi r) = \inf_{r\in\mathbb{N}} \max(r, b, c_2, \phi r) = \max(b, c_2)
\end{aligned}$$

since all these infima are attained for $z = r = s = 0$. This shows (*), and so $\phi\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow 1$ is a flat module.

On the other hand, ϕ is not an **FSW**-ideal: we have $\phi(2) < 2$ and $\phi(3) < 2$ but there is no $z \in \mathbb{N}$ with $\phi(z) < 1$ and $\mathbb{N}(2,z) < 2$ and $\mathbb{N}(3,z) < 2$.

4.4. **The absolute case.** For any quantale $\mathcal{Q}$, we can consider Φ being the empty class and therefore $JX = \widehat{X}$ is the collection of all modules of type $X \relbar\joinrel\mapstochar\joinrel\rightarrow 1$. In this case, every cocontinuous $\mathcal{Q}$-functor $\alpha\colon X \to \mathcal{Q}$ is an open module. Furthermore, every totally continuous cocomplete $\mathcal{Q}$-category is open module determined since $\Downarrow_X(x,-)\colon X \to \mathcal{Q}$ is in FX. Finally, a $\mathcal{Q}$-functor $f\colon X \to Y$ reflects open modules if and only if f is left adjoint. Therefore Theorem 3.9 states that the category of totally continuous cocomplete $\mathcal{Q}$-categories and left adjoint $\mathcal{Q}$-functors is self-dual.

4.5. **A somehow different example.** We consider now $\mathcal{Q} = [0,\infty]$ where $\otimes = +$, with the class J of modules described in Example 2.13. However, for technical reasons we consider the unique module $\varnothing \relbar\joinrel\mapstochar\joinrel\rightarrow 1$ as a formal ball, so that $J\varnothing = 1$. Consequently, the empty space is not J-cocomplete. We will show now that our duality theorem holds in this case too, despite the fact that this class of modules is (to our knowledge) not defined via a class of limit weights.

Let now X be a J-cocomplete and J-continuous metric space. We write $\Downarrow\colon X \to JX$ for the left adjoint to $\mathsf{S}\colon JX \to X$. Hence, for any $x \in X$, $\Downarrow(x) \in JX$ is of the form $\Downarrow(x) = X(-,x_1) + u$ for some $x_1 \in X$ and $u \in [0,\infty]$. Note that $u < \infty$ if x is not the bottom element of X. Assume that $\Downarrow(x_1) = X(-,x_2) + u_2$. Then

$$X(-,x_1) + u = \Downarrow(x) = \Downarrow(x_1 + u_1) = \Downarrow(x_1) + u_1 = X(-,x_2) + u_2 + u_1,$$

hence, $X(-,x_1) = X(-,x_2)+u_2$. In particular, $0 = X(x_1,x_2)+u_2$, and therefore $u_2 = 0$ and we obtain $\Downarrow(x_1) = \mathsf{y}(x_1)$. Let A be the equaliser of y and $\Downarrow$, that is, $A = \{x \in X \mid \Downarrow(x) = \mathsf{y}(x)\}$. By the considerations above, $\Downarrow : X \to JX$ factors through the inclusion $JA \hookrightarrow JX$. Moreover, for any $X(-,x)+u$ with $x \in A$, $\Downarrow(x+u) = \Downarrow(x)+u = X(-,x)+u$, which gives $X \cong JA$. We also remark that $x \in A$ if and only if $X(x,-) : X \to [0,\infty]$ preserves tensoring. One has $\phi \in FX$ precisely if $\phi = X(x,-)+u$ for some $x \in X$ and $u \in [0,\infty]$ and if, moreover, ϕ preserves tensoring. If $u < \infty$, then also $X(x,-)$ preserves tensoring, hence $x \in A$. Consequently, $FX \cong J(A^{\mathrm{op}})$.

Consider now $f : X \to Y$ with $X \cong JA$ and $Y \cong JB$ as above. Then f is open module reflecting if, and only if, for each $y_0 \in B$, there exists some $x_0 \in A$ and some $v \in [0,\infty]$ with $Y(y_0,f(-)) = X(x_0,-)+v$. We show that f necessarily preserves tensoring. To this end, let $x \in X$ and $u \in [0,\infty]$. Then

$$Y(y_0,f(x+u)) = X(x_0,x+u)+v = X(x_0,x)+v+u = Y(y_0,f(x))+u = Y(y_0,f(x)+u)$$

for all $y_0 \in B$, hence $f(x+u) = f(x)+u$. Therefore f corresponds to a module $\phi : B \relbar\joinrel\mapstochar\joinrel\rightarrow A$ in the sense that, when identifying X with JA and Y with JB, then $f(\psi) = \psi \cdot \phi$. Hence, for any $x \in A$, $x^* \cdot \phi = \phi(-,x)$ belongs to JB, and the f being open module reflecting translates to $\phi \cdot y_* = \phi(y,-) \in J(A^{\mathrm{op}})$ for all $y \in B$. Recall that for each module $\phi : B \relbar\joinrel\mapstochar\joinrel\rightarrow A$ we have its dual $\phi^{\mathrm{op}} : A^{\mathrm{op}} \relbar\joinrel\mapstochar\joinrel\rightarrow B^{\mathrm{op}}$, $\phi^{\mathrm{op}}(x,y) = \phi(y,x)$, and with this notation the latter condition reads as $y^* \cdot \phi^{\mathrm{op}} \in J(A^{\mathrm{op}})$ for all $y \in B^{\mathrm{op}}$. We conclude that the category of J-cocomplete and J-continuous metric spaces and open module reflecting contraction maps is dually equivalent to the category of all metric spaces with morphisms those modules $\phi : X \to Y$ satisfying

$$\forall y \in Y\,.\,(y^* \cdot \phi \in JX) \qquad \text{and} \qquad \forall x \in X^{\mathrm{op}}\,.\,(x^* \cdot \phi^{\mathrm{op}} \in J(Y^{\mathrm{op}})),$$

and the latter category is obviously self-dual.

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