Lawvere completeness in Topology

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Abstract

It is known since 1973 that Lawvere's notion of Cauchy-complete enriched category is meaningful for metric spaces: it captures exactly Cauchy-complete metric spaces. In this paper, we introduce the corresponding notion of Lawvere completeness for $(\mathbb{T},\mathsf{V})$-categories and show that it has an interesting meaning for topological spaces and quasi-uniform spaces: for the former ones it means weak sobriety while for the latter it means Cauchy completeness. Further, we show that V has a canonical $(\mathbb{T},\mathsf{V})$-category structure which plays a key role: it is Lawvere-complete under reasonable conditions on the setting; this structure permits us to define a Yoneda embedding in the realm of $(\mathbb{T},\mathsf{V})$-categories.

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0 Introduction

The notion of $(\mathbb{T},\mathsf{V})$-category – as introduced in [17] under the name $(\mathbb{T},\mathsf{V})$-algebra – is simultaneously a generalisation of V-enriched category and the lax version of Eilenberg-Moore $\mathbb{T}$-algebra introduced in [1, 9]. The latter paper had its origins in the observation that special classes of morphisms of the category Top of topological spaces, namely triquotient, effective descent and exponentiable maps, could be described using ultrafilter convergence, with the ultrafilter monad playing a key role in their study (see [7, 8, 14]). Replacing the ultrafilter monad by a general monad $\mathbb{T}$, and considering a commutative and unital quantale V, $(\mathbb{T},\mathsf{V})$-categories and functors constitute a topological category and several results obtained in Top using ultrafilter convergence could be generalised to this setting (cf. [11, 15, 12, 23]). On the other hand, thinking of $(\mathbb{T},\mathsf{V})$-categories as generalised V-categories, one can import notions and techniques of V-enriched categories into this setting. This paper follows this line of research.

Lawvere in his 1973 paper [28] formulates a notion of *complete* V-category and shows that for (generalised) metric spaces it means Cauchy completeness. This notion of completeness is now well-known in the categorical community, particularly in the context of Ab-enriched categories, under the name *Freyd-Karoubi complete category.* However, it never caught the attention of the topological community. In this paper, we extend Lawvere's notion of complete V-category to

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the topological setting of $(\mathbb{T},\mathsf{V})$-categories, and show that it encompasses well-known notions in topological categories, meaning *weakly sober space* for topological spaces, *weakly sober approach space* for approach spaces, and *Cauchy-complete* for quasi-uniform spaces.

We also present a first step towards a possible construction of completion. Indeed, in the setting of V-categories, it is well-known that the completion of a V-category may be obtained from the Yoneda embedding $X \to \mathsf{V}^{X^{\mathrm{op}}}$. As in the V-setting, in the $(\mathbb{T},\mathsf{V})$-setting V has a canonical $(\mathbb{T},\mathsf{V})$-categorical structure and every $(\mathbb{T},\mathsf{V})$-category X has a canonical dual X^{op}. Using this structure and the free Eilenberg-Moore algebra structure $|X|$ on TX, we get two "Yoneda-like" morphisms

$$X \to \mathsf{V}^{X^{\mathrm{op}}} \ \text{ and } \ X \to \mathsf{V}^{|X|}.$$

For the latter one we prove a Yoneda Lemma (see 4.2).

Furthermore, we show that, under suitable conditions, V is a Lawvere-complete $(\mathbb{T},\mathsf{V})$-category, a first step towards a completion construction which will be the subject of a forthcoming paper [26].

Besides the contribution to the establishment of the notion of Lawvere completeness in topological settings, our study contributes to the categorical development of $(\mathbb{T},\mathsf{V})$-categories, as it presents both the theory of bimodules and the Yoneda lemma in this setting. These notions are proving to be very useful in further work, in particular in the study of injectivity [25, 13].

In order to motivate the notions and results of the paper, in Section 1 we recall the notions and properties of V-categories we will generalise throughout. First we introduce V-categories and V-bimodules, and define Lawvere-complete V-categories, for a commutative and unital quantale V. V is then naturally equipped with the V-categorical structure hom. We give a direct proof of Lawvere completeness of the V-category $(\mathsf{V},\hom)$.

In Section 2 we describe our basic setting for the study of $(\mathbb{T},\mathsf{V})$-categories. We describe Kleisli convolution in the category $\mathsf{V}\text{-}\mathsf{Rel}$ of V-valued relations and define $(\mathbb{T},\mathsf{V})$-bimodules. Although $(\mathbb{T},\mathsf{V})$-bimodules do not compose in general, one can still formulate and study the notion of Lawvere-complete $(\mathbb{T},\mathsf{V})$-categories.

Similarly to what was done for V-categories, we define a canonical $(\mathbb{T},\mathsf{V})$-categorical structure on V, as the composition of hom with the canonical $\mathbb{T}$-algebra structure on V described by Manes in [31]. This is the subject of Section 3. In addition we also prove that, under some conditions, the $(\mathbb{T},\mathsf{V})$-category V is Lawvere-complete.

In Section 4 we present the Yoneda embedding for V-categories as a byproduct of the fact that a V-relation $\psi : X \mathrel{-\!\!\!\mapstochar\!\!\rightarrow} Y$ between V-categories (X,a) and (Y,b) is a V-bimodule if and only if, as a map, $\psi : X^{\mathrm{op}}\otimes Y \to \mathsf{V}$ is a V-functor (Theorem 1.5); then the monoidal-closed structure of $\mathsf{V}\text{-}\mathsf{Cat}$ gives us the *Yoneda functor* $X \to \mathsf{V}^{X^{\mathrm{op}}}$. In the $(\mathbb{T},\mathsf{V})$-setting this construction becomes more elaborate (see Theorem 3.3): a V-relation $\psi : TX \mathrel{-\!\!\!\mapstochar\!\!\rightarrow} Y$ is a $(\mathbb{T},\mathsf{V})$-bimodule $\psi : (X,a) \mathrel{-\!\!\circ\!\!\rightarrow} (Y,b)$ if and only if both $\psi : |X| \otimes Y \to \mathsf{V}$ and $\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$ are $(\mathbb{T},\mathsf{V})$-functors. Thus, given a $(\mathbb{T},\mathsf{V})$-category (X,a), the $(\mathbb{T},\mathsf{V})$-bimodule $a : X \mathrel{-\!\!\circ\!\!\rightarrow} X$ gives rise to two *Yoneda* $(\mathbb{T},\mathsf{V})$*-functors* $X \to \mathsf{V}^{X^{\mathrm{op}}}$ and $X \to \mathsf{V}^{|X|}$.

In Section 5 we present the announced topological examples, with the exception of quasi-uniform spaces, which are presented in the last section, due to the fact that their presentation as lax algebras does not fit in the $(\mathbb{T},\mathsf{V})$-setting (as shown in [30]).

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1 A primer on V-categories

Although the material of this section can be found essentially in [27], we find that its inclusion here may enlighten the corresponding – but more technical – notions and results for $(\mathbb{T},\mathsf{V})$-categories presented in the forthcoming sections.

1.1 V**.** Throughout, V is a (commutative and unital) *quantale*. In other words, V is a complete lattice equipped with a symmetric and associative tensor product $\otimes$, with unit k, and with right adjoint hom; that is, for each $u,v,w\in\mathsf{V}$,

$$u\otimes v\le w \iff v\le \hom(u,w).$$

Considered as a (thin) category, V is said to be *symmetric monoidal-closed.* If k is the bottom element $\perp$ of V, then $\mathsf{V}=1$ is the trivial lattice. Throughout this paper *we assume that* V *is non-trivial, i.e.* $k\neq\perp$.

Every non-trivial Heyting algebra – with $\otimes=\wedge$ and $k=\top$ the top element – is an example of such a lattice, in particular the two-element chain $\mathsf{2}=\{\text{false}\models\text{true}\}$, with the monoidal structure given by "&" (and) and "true". The extended real half-line $\mathsf{P}=[0,\infty]$, with the categorical structure induced by the relation $\geq$ (i.e., $a\to b$ means $a\geq b$), admits several interesting monoidal structures. First of all, with $\otimes=\max$ it is a Heyting algebra $\mathsf{P}_{\max}$. Another possible choice for $\otimes$ is $+$; we denote P equipped with this tensor by P_+. Note that in this example the right adjoint hom is given by truncated minus: $\hom(u,v)=\max\{v-u,0\}$.

1.2 V**-**Rel**.** The category V-Rel of V-*relations* [3, 17] has sets as objects, and a morphism $r:X\to\!\!\!\!\!\mapsto Y$ in V-Rel is a map $r:X\times Y\to\mathsf{V}$. Composition of V-relations $r:X\to\!\!\!\!\!\mapsto Y$ and $s:Y\to\!\!\!\!\!\mapsto Z$ is defined as relational multiplication:

$$s\cdot r(x,z)=\bigvee_{y\in Y} r(x,y)\otimes s(y,z).$$

The identity arrow $1_X:X\to\!\!\!\!\!\mapsto X$ in V-Rel is the V-relation which sends all diagonal elements (x,x) to k and all other elements to the bottom element $\perp$ of V. In fact, each Set-map $f:X\to Y$ can be interpreted as the V-relation

$$f:X\to\!\!\!\!\!\mapsto Y,\ f(x,y)=\begin{cases}k & \text{if } f(x)=y,\\ \perp & \text{else.}\end{cases}$$

To keep notation simple, in the sequel we will write $f:X\to Y$ rather than $f:X\to\!\!\!\!\!\mapsto Y$ for a V-relation induced by a map. The formula for relational composition becomes considerably easier if one of the V-relations is a Set-map:

$$s \cdot f(x,z) = s(f(x),z), \qquad\qquad g \cdot r(x,z) = \bigvee_{y \in g^{-1}(z)} r(x,y)$$

for maps $f : X \to Y$ and $g : Y \to Z$ and V-relations $r : X \rightarrow\!\!\!\!\!\mapsto Y$ and $s : Y \rightarrow\!\!\!\!\!\mapsto Z$.

The complete order on V induces a complete order on $\mathsf{V}\text{-}\mathsf{Rel}(X,Y) = \mathsf{V}^{X\times Y}$: for V-relations $r, r' : X \rightarrow\!\!\!\!\!\mapsto Y$ we define

$$r \le r' \;:\iff\; \forall x \in X\ \forall y \in Y\ r(x,y) \le r'(x,y).$$

The *transpose* $r^\circ : Y \rightarrow\!\!\!\!\!\mapsto X$ of a V-relation $r : X \rightarrow\!\!\!\!\!\mapsto Y$ is defined by $r^\circ(y,x) = r(x,y)$. It is easy to see that $(-)^\circ : \mathsf{V}\text{-}\mathsf{Rel}(X,Y) \to \mathsf{V}\text{-}\mathsf{Rel}(Y,X)$ is order-preserving, and

$$1_X^\circ = 1_X, \qquad\qquad (s \cdot r)^\circ = r^\circ \cdot s^\circ, \qquad\qquad r^{\circ\circ} = r.$$

For each Set-map $f : X \to Y$ we have $1_X \le f^\circ \cdot f$ and $f \cdot f^\circ \le 1_Y$, i.e. f is left adjoint to f° and we write $f \dashv f^\circ$. In general, given V-relations $r : X \rightarrow\!\!\!\!\!\mapsto Y$ and $s : Y \rightarrow\!\!\!\!\!\mapsto X$, we say that r is left adjoint to s (and that s is right adjoint to r) if $1_X \le s \cdot r$ and $1_Y \ge r \cdot s$.

Lemma. *Let V be a quantale and $r, r' : X \rightarrow\!\!\!\!\!\mapsto Y$ and $s, s' : Y \rightarrow\!\!\!\!\!\mapsto X$ be V-relations such that $r \dashv s$ and $r' \dashv s'$. Then $r \le r'$ if and only if $s' \le s$; consequently, if $r \le r'$ and $s \le s'$, then $r = r'$ and $s = s'$.*

As a consequence of the lemma above we have that left and right adjoints are uniquely determined. Therefore we say that r is left adjoint if it has a right adjoint s, and likewise, s is right adjoint if it has a left adjoint r. In pointwise notation, we have $r \dashv s$ if and only if

$$\forall x \in X\ \bigvee_{y \in Y} r(x,y) \otimes s(y,x) \ge k,$$

$$\forall x \in X\ \forall y, y' \in Y\ s(y,x) \otimes r(x,y') \le \begin{cases} \bot & \text{if } y \ne y', \\ k & \text{if } y = y' \end{cases}$$

which, by symmetry of $\otimes$, is equivalent to

$$\forall x \in X\ \bigvee_{y \in Y} r(x,y) \otimes s(y,x) = k,$$

$$\forall x \in X\ \forall y, y' \in Y\ (y \ne y' \;\Rightarrow\; s(y,x) \otimes r(x,y') = \bot).$$

Our next example shows that there exist indeed left adjoint V-relations which are not induced by Set-maps.

Example. Consider a set X and the Boolean algebra $\mathsf{V} = PX$ the powerset of X. Define a V-relation $r : 1 \rightarrow\!\!\!\!\!\mapsto X$ by putting $r(\star, x) = \{x\}$ for $x \in X$. Then

$$r^\circ \cdot r(\star,\star) = \bigcup_{x \in X} \{x\} = X \qquad \text{and} \qquad r \cdot r^\circ(x,y) = \{x\} \cap \{y\} = \begin{cases} \emptyset & \text{if } x \ne y, \\ \{x\} & \text{if } x = y, \end{cases}$$

hence $r \dashv r^\circ$. But r is not a Set-map unless X has at most one element.

We wish to characterise those quantales V for which the class of left adjoint V-relations coincides with the class of Set-maps. In order to do so we introduce some notation. Let $u, v \in \mathsf{V}$. We say that v is a $\otimes$-*complement* of u if

$$u \vee v = k \qquad\qquad \text{and} \qquad\qquad u \otimes v = \bot.$$

Each $u \in \mathsf{V}$ has at most one $\otimes$-complement. Moreover, if u is $\otimes$-complemented (i.e. has a $\otimes$-complement v), then

$$u = u \otimes k = u \otimes (u \vee v) = (u \otimes u) \vee (u \otimes v) = u \otimes u,$$

that is, u is idempotent. Our next result generalises [21, 2.14].

Proposition. *Let* V *be a quantale. Each left adjoint* V*-relation is a* Set*-map if and only if* k *and* $\bot$ *are the only* $\otimes$*-complemented elements of* V *and*

$$\forall u, v \in \mathsf{V}\ (u \otimes v = k \ \ \Rightarrow \ \ u = k = v).$$

Proof. Assume first that each left adjoint V-relation is a Set-map. Let $u, v \in \mathsf{V}$. If $u \otimes v = k$, then $u \dashv v$, for u, v seen as relations, and we have $u = v = k$. Suppose that $u \vee v = k$ and $u \otimes v = \bot$. Let $X = \{u, v\}$ and define $r : 1 \relbar\joinrel\mapsto X$ with $r(\star, u) = u$ and $r(\star, v) = v$. Then $r \dashv r^\circ$ and, by assumption, $u = k$ or $v = k$.
Let $r : X \relbar\joinrel\mapsto Y$ and $s : Y \relbar\joinrel\mapsto X$ be V-relations such that $r \dashv s$. Let $x \in X$. There is some $y \in Y$ such that $r(x, y) \otimes s(y, x) > \bot$. Then

$$k = (r(x, y) \otimes s(y, x)) \vee \bigvee_{y' \neq y} (r(x, y') \otimes s(y', x))$$

and

$$r(x, y) \otimes s(y, x) \otimes \bigvee_{y' \neq y} r(x, y') \otimes s(y', x) = \bigvee_{y' \neq y} r(x, y) \otimes s(y, x) \otimes r(x, y') \otimes s(y', x) = \bot.$$

Hence, by assumption, $r(x, y) = k = s(y, x)$ and $r(x, y') \otimes s(y', x) = \bot$ for all $y' \neq y$. We have shown that, for each $x \in X$, there exists exactly one $y \in Y$ with $r(x, y) = k = s(y, x)$. Consider now $f : X \to Y$ which assigns to x this unique y. Clearly, $f \leq r$, but also $f^\circ \leq s$ since

$$f^\circ(y, x) = k \ \ \Rightarrow \ \ f(x) = y \ \ \Rightarrow \ \ s(y, x) = k.$$

The assertion follows now from the previous lemma. $\square$

1.3 V-categories. V-enriched categories were introduced and studied in [18, 27] in the more general context of symmetric monoidal-closed categories. For a very nice presentation of this material we refer to [28]. In the next subsections we recall some well-known facts about V-categories, which will serve as a guideline for our study of $(\mathbb{T}, \mathsf{V})$-categories.

A V-*enriched category* (or simply V-*category*) is a pair (X, a) with X a set and $a : X \relbar\joinrel\mapsto X$ a V-relation such that

$$1_X \leq a \quad \text{and } a \cdot a \leq a$$

(hence $a = a \cdot a$); equivalently, the map $a : X \times X \to \mathsf{V}$ satisfies the following conditions:

(R) for each $x \in X$, $k \leq a(x,x)$;

(T) for each $x, x', x'' \in X$, $a(x,x') \otimes a(x',x'') \leq a(x,x'')$.

Given V-categories (X,a) and (Y,b), a V-*functor* $f : (X,a) \to (Y,b)$ is a map $f : X \to Y$ such that, for each $x, x' \in X$, $a(x,x') \leq b(f(x), f(x'))$. V-categories and V-functors are the objects and morphisms of the category $\mathsf{V}\text{-}\mathsf{Cat}$. Finally, given a V-category $X = (X,a)$, the *dual category* X^{op} of X is defined by $X^{\mathrm{op}} = (X, a^\circ)$.

We remark that $\mathsf{V}\text{-}\mathsf{Cat}$ is actually a *closed category* since the tensor product on V can be naturally transported to $\mathsf{V}\text{-}\mathsf{Cat}$. More precisely, for V-categories $X = (X,a)$ and $Y = (Y,b)$, we put $X \otimes Y = (X \times Y, a \otimes b)$ where $a \otimes b((x,y),(x',y')) = a(x,x') \otimes b(y,y')$ for all $x, x' \in X$ and $y, y' \in Y$. Then, for each V-category $X = (X,a)$, the functor $X \otimes (-) : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Cat}$ has a right adjoint $(-)^X$ defined by $Y^X = (\mathsf{V}\text{-}\mathsf{Cat}(X,Y), d)$ with $d(f,g) = \bigwedge_{x \in X} b(f(x), g(x))$.

Being monoidal-closed, V has a natural structure as a V-category:

$$\hom : \mathsf{V} \relbar\joinrel\mapstochar\joinrel\rightarrow \mathsf{V}.$$

Indeed, for $u, v, w \in \mathsf{V}$,

$$k \otimes v = v \;\Rightarrow\; k \leq \hom(v,v),$$

$$u \otimes (\hom(u,v) \otimes \hom(v,w)) \leq v \otimes \hom(v,w) \leq w \;\Rightarrow\; \hom(u,v) \otimes \hom(v,w) \leq \hom(u,w),$$

that is, $1_{\mathsf{V}} \leq \hom$ and $\hom \cdot \hom \leq \hom$.

For $\mathsf{V} = 2$, with the usual notation $x \leq x' :\iff a(x,x') = \text{true}$, axioms (R) and (T) read as

$$\forall x \in X \;\text{true} \models x \leq x \qquad \text{and} \qquad \forall x, x', x'' \in X \; x \leq x' \;\&\; x' \leq x'' \models x \leq x'',$$

that is, $(X, \leq)$ is an ordered set. (Note that we do *not* require $\leq$ to be anti-symmetric.) A 2-functor is a map $f : (X, \leq) \to (Y, \leq)$ between ordered sets such that

$$\forall x, x' \in X \; x \leq x' \;\models\; f(x) \leq f(x');$$

that is, f is a monotone map. Hence $2\text{-}\mathsf{Cat}$ is equivalent to the category Ord of ordered sets and monotone maps.

A $\mathsf{P}_{\!+}$-category is a set X endowed with a map $a : X \times X \to \mathsf{P}_{\!+}$ such that

$$\forall x \in X \; 0 \geq a(x,x) \qquad \text{and} \qquad \forall x, x', x'' \in X \; a(x,x') + a(x',x'') \geq a(x,x'');$$

that is, $a : X \times X \to \mathsf{P}_{\!+}$ is a metric on X. A $\mathsf{P}_{\!+}$-functor is a map $f : (X,a) \to (Y,b)$ between metric spaces satisfying the following inequality:

$$\forall x, x' \in X \; a(x,x') \geq b(f(x), f(x')),$$

which means precisely that f is a non-expansive map. Therefore the category $\mathsf{P}_{\!+}\text{-}\mathsf{Cat}$ coincides with the category Met of metric spaces and non-expansive maps. (For more details, see [28, 17].)

For $\mathsf{V} = \mathsf{P}_{\!\max}$, the transitivity axiom (T) reads as

$$\max\{a(x,x'), a(x',x'')\} \geq a(x,x''),$$

hence the category $\mathsf{P}_{\!\max}\text{-}\mathsf{Cat}$ coincides with the category UMet of *ultrametric spaces and non-expansive maps.*

1.4 V-bimodules. Given V-categories (X,a) and (Y,b), a *bimodule* (or *profunctor*, or *distributor* – see [2, 5, 36]) $\psi:(X,a){\circ\!\!\!\longrightarrow}(Y,b)$ is a V-relation $\psi:X{\longrightarrow\!\!\!\!\!\!\!\mapsto}\,Y$ such that $\psi\cdot a\le\psi$ and $b\cdot\psi\le\psi$; that is, for each $x,x'\in X$ and $y,y'\in Y$,

$$a(x,x')\otimes\psi(x',y)\le\psi(x,y)\qquad\text{and}\qquad\psi(x,y')\otimes b(y',y)\le\psi(x,y).$$

It is easy to verify that bimodules compose and that V-categorical structures are themselves bimodules. In fact, they are the identities for the composition of bimodules, that is, for any bimodule $\psi:(X,a){\circ\!\!\!\longrightarrow}(Y,b)$, $\psi\cdot a=\psi$ and $b\cdot\psi=\psi$. Therefore, V-categories and V-bimodules constitute a category, which we will denote by V-Mod. The category V-Mod inherits the bicategorical structure of V-Rel via the forgetful functor V-Mod $\to$ V-Rel.

1.5 V-functors as V-bimodules. Any V-functor $f:(X,a)\to(Y,b)$ defines a pair of relations $f_*:(X,a){\longrightarrow\!\!\!\!\!\!\!\mapsto}\,(Y,b)$ and $f^*:(Y,b){\longrightarrow\!\!\!\!\!\!\!\mapsto}\,(X,a)$, with $f_*=b\cdot f$ and $f^*=f^\circ\cdot b$, that is $f_*(x,y)=b(f(x),y)$ and $f^*(y,x)=b(y,f(x))$, which are in fact bimodules: for every $x,x'\in X$ and $y,y'\in Y$,

$$\begin{aligned}a(x,x')\otimes f_*(x',y)&=a(x,x')\otimes b(f(x'),y)\le b(f(x),f(x'))\otimes b(f(x'),y)\le b(f(x),y),\\ f_*(x,y')\otimes b(y'y)&=b(f(x),y')\otimes b(y',y)\le b(f(x),y),\end{aligned}$$

and similarly for f^*.

Moreover, the bimodules f_* and f^* form an adjunction, as we show next. We recall first that, given bimodules $\varphi:(X,a){\circ\!\!\!\longrightarrow}(Y,b)$ and $\psi:(Y,b){\circ\!\!\!\longrightarrow}(X,a)$, φ is left adjoint to ψ, $\varphi\dashv\psi$, if $1_{(X,a)}\le\psi\cdot\varphi$ and $\varphi\cdot\psi\le 1_{(Y,b)}$, i.e. $a\le\psi\cdot\varphi$ and $\varphi\cdot\psi\le b$. It is now straightforward to check that $f_*\dashv f^*$, since, for $x,x'\in X$ and $y,y'\in Y$, the inequality

$$a(x,x')\le\bigvee_{y\in Y}f_*(x,y)\otimes f^*(y,x')=\bigvee_{y\in Y}b(f(x),y)\otimes b(y,f(x'))=b(f(x),f(x'))$$

follows from V-functoriality of f, while

$$\bigvee_{x\in X}f^*(y,x)\otimes f_*(x,y')=\bigvee_{x\in X}b(y,f(x))\otimes b(f(x),y')\le b(y,y')$$

follows from the transitivity axiom (T) for V-categories. A quite different connection between functors and bimodules offers the following

Theorem. *For V-categories $X=(X,a)$ and $Y=(Y,b)$ and a V-relation $\psi:X{\longrightarrow\!\!\!\!\!\!\!\mapsto}\,Y$, the following conditions are equivalent:*

(i) *$\psi:X{\circ\!\!\!\longrightarrow}Y$ is a bimodule;*

(ii) *$\psi:X^{\mathrm{op}}\otimes Y\to\mathsf{V}$ is a V-functor.*

Proof. (i) $\Rightarrow$ (ii): For $x,x'\in X$ and $y,y'\in Y$,

$$\begin{aligned}\psi(x,y)\otimes a^\circ(x,x')\otimes b(y,y')&=a(x',x)\otimes\psi(x,y)\otimes b(y,y')\\&\le\psi(x',y)\otimes b(y,y')\\&\le\psi(x',y'),\end{aligned}$$

hence

$$a^\circ(x,x') \otimes b(y,y') \le \hom(\psi(x,y),\psi(x',y')).$$

(ii) $\Rightarrow$ (i): For $x, x' \in X$ and $y, y' \in Y$,

$$\begin{array}{rcl} a(x,x') \otimes \psi(x',y) & \le & \psi(x',y) \otimes a^\circ(x',x) \otimes b(y,y) \\ & \le & \psi(x,y), \end{array}$$

that is $\psi \cdot a \le \psi$, and

$$\begin{array}{rcl} \psi(x,y') \otimes b(y',y) & \le & \psi(x,y') \otimes a^\circ(x,x) \otimes b(y',y) \\ & \le & \psi(x,y), \end{array}$$

that is $b \cdot \psi \le \psi$. $\square$

Corollary (Yoneda lemma)**.** *There is a* V*-functor* $\ulcorner a \urcorner : X \to \mathsf{V}^{X^{\mathrm{op}}}$*. Moreover, for each* $x \in X$ *and* $f \in \mathsf{V}^{X^{\mathrm{op}}}$*, we have*

$$d(a(-,x),f) = f(x)$$

(where d *is the structure of* V*-category on* $V^{X^{\mathrm{op}}}$ *defined in* 1.3*).*

Proof. Note that $d(a(-,x),f) = \bigwedge\limits_y \hom(a(y,x),f(y)) \le f(x)$. On the other hand, for each $y \in X$,

$$\begin{array}{rcl} a(y,x) \le \hom(f(x),f(y)) & \iff & f(x) \otimes a(y,x) \le f(y) \\ & \iff & f(x) \le \hom(a(y,x),f(y)). \end{array}$$

$\square$

1.6 Lawvere-complete V-categories.

Definition. A V-category (X,a) is said to be *Lawvere-complete* if, for any V-category (Y,b) and every pair of adjoint bimodules

$$\begin{tikzcd} Y \arrow[loop left, "b"] \arrow[r, bend left, "\varphi", "\circ" marking] \arrow[r, phantom, "\perp"] & X \arrow[l, bend left, "\psi", "\circ" marking] \arrow[loop right, "a"] \end{tikzcd}$$

φ is in the image of $(\ )_* : \mathsf{V}\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Mod}$, i.e. there exists a V-functor $f : (X,a) \to (Y,b)$ such that $f_* = \varphi$ and $f^* = \psi$. In this situation one says that f *represents the adjunction* $\varphi \dashv \psi$, and thus *the adjunction is representable.*

It is interesting to notice that, in order to check Lawvere completeness, we can restrict ourselves to the case where (Y,b) is the V-category $(1,p)$, where $1 = \{\star\}$ is a singleton and $p(\star,\star) = k$. If an adjunction is represented by a V-functor $f : (1,p) \to (X,a)$, $f(\star)$ is said to *represent* the adjunction.

Proposition. *For a* V*-category* (X,a)*, the following conditions are equivalent:*

(i) (X,a) *is Lawvere-complete;*

(ii) *for each pair of adjoint bimodules* $(\varphi : (1,p) \multimap (X,a)) \dashv (\psi : (X,a) \multimap (1,p))$*, there exists a* V*-functor* $f : (1,p) \to (X,a)$ *such that* $\varphi = f_*$ *and* $\psi = f^*$*.*

Proof. It is a special case of Proposition 2.7. □

Theorem. *The* V*-category* $(\mathsf{V}, \hom)$ *is Lawvere-complete.*

We omit the proof of this theorem because it is a particular case of Theorem 3.4.

A new insight on Lawvere completeness for V-categories may be found in [35, 26].

2 Modules and Lawvere completeness in the $(\mathbb{T}, \mathsf{V})$-setting

In the first part of this section we present the setting for the study of $(\mathbb{T}, \mathsf{V})$-categories, or lax (Eilenberg-Moore) algebras, that are studied in more detail in [9, 17, 10].

2.1 $\mathbb{T}$ and its extension. Recall that a *monad* $\mathbb{T} = (T, e, m)$ on Set consists of a functor $T : \mathsf{Set} \to \mathsf{Set}$ together with natural transformations $e : \mathrm{Id}_{\mathsf{Set}} \to T$ and $m : TT \to T$ such that

$$m \cdot Tm = m \cdot m_T \qquad \text{and} \qquad m \cdot Te = 1_T = m \cdot e_T$$

(the natural transformation e is the *unit* of the monad, while m is its *multiplication*). There are two *trivial monads* on Set, one sending all sets X to the terminal set 1, and the other with $T\varnothing = \varnothing$ and $TX = 1$ for $X \neq \varnothing$. Any other monad is called *non-trivial.*

By a *lax extension* of a Set-monad $\mathbb{T} = (T, e, m)$ to V-Rel we mean an extension of the endofunctor $T : \mathsf{Set} \to \mathsf{Set}$ to V-relations acting on Set-maps as T and satisfying

(a) $(Ta)^\circ = T(a^\circ)$ (and we write Ta°),

(b) $Tb \cdot Ta \leq T(b \cdot a)$,

(c) $a \leq a' \ \Rightarrow\ Ta \leq Ta'$,

(d) $e_Y \cdot a \leq Ta \cdot e_X$,

(e) $m_Y \cdot T^2 a \leq Ta \cdot m_X$,

for all $a, a' : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and $b : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$. (The conditions for our extension are stronger than Seal's in [34].). Note that we have automatically equality in (b) if $a = f$ is a Set-map. A Set-monad $\mathbb{T} = (T, e, m)$ admitting a lax extension to V-Rel is called V*-admissible.* Although $\mathbb{T}$ may have many lax extensions to V-Rel, in the sequel we usually have a fixed extension in mind when talking about a V-admissible monad. Trivially, the identity monad $\mathbb{1} = (\mathrm{Id}, 1, 1)$ on Set can be extended to the identity monad on V-Rel. In [1] M. Barr shows how to extend Set-monads to $\mathsf{Rel} = \mathsf{2}$-$\mathsf{Rel}$: first observe that each relation $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ can be written as $r = p \cdot q^\circ$ where $q : R \to X$ and $p : R \to Y$ are the projection maps, then put $Tr = Tp \cdot Tq^\circ$. All conditions above but the second one are satisfied, and this extension satisfies (b) if and only if the Set-functor T has the *Beck-Chevalley property* (BC) (that is, sends pullbacks to weak pullbacks). In [10] we showed how to make the step from Rel to V-Rel, provided that in addition V is *constructively completely distributive (ccd)* (that is, if $\bigvee : 2^{Y^{\mathrm{op}}} \to Y$ has a left adjoint; for more details see [37].). Given a monad $\mathbb{T} = (T, e, m)$ and a V-relation $a : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, we define relations $a_v : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ $(v \in \mathsf{V})$ by $a_v(x, y) = \text{true} \iff a(x, y) \geq v$ and put, for $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$,

$$Ta(\mathfrak{x}, \mathfrak{y}) = \bigvee\{v \in \mathsf{V} \mid Ta_v(\mathfrak{x}, \mathfrak{y}) = \text{true}\}.$$

Then the formula above defines an extension of the functor $T : \mathsf{Set} \to \mathsf{Set}$ provided that either $k = \top$ or $T\varnothing = \varnothing$. In both cases it gives a lax extension of the monad, i.e. conditions (a)-(e) above are satisfied. In addition we have

(f) $Tb \cdot Ta = T(b \cdot a)$ provided that $\otimes = \wedge$,

(g) $Tg \cdot Ta = T(g \cdot a)$,

for all V-relations $a : X \,{\longrightarrow\!\!\!\!\!\mapsto}\, Y$ and $b : Y \,{\longrightarrow\!\!\!\!\!\mapsto}\, Z$ and all maps $g : Y \to Z$. In some occasions we will need that the (Set-based) natural transformation $m : TT \to T$ has (BC) (that is, each naturality square is a weak pullback); this guarantees that m is also a natural transformation for the extension of T to V-Rel described above.

2.2 $(\mathbb{T}, \mathsf{V})$-categories. Let $\mathbb{T} = (T, e, m)$ be a V-admissible monad. A $(\mathbb{T}, \mathsf{V})$-*category* is a pair (X, a) consisting of a set X and a V-relation $a : TX \,{\longrightarrow\!\!\!\!\!\mapsto}\, X$ such that:

$$1_X \leq a \cdot e_X \quad \text{and } a \cdot Ta \leq a \cdot m_X;$$

that is, the map $a : TX \times X \to \mathsf{V}$ satisfies the conditions:

(R) for each $x \in X$, $k \leq a(e_X(x), x)$;

(T) for each $\mathfrak{X} \in T^2X$, $\mathfrak{x} \in TX$, $x \in X$, $Ta(\mathfrak{X}, \mathfrak{x}) \otimes a(\mathfrak{x}, x) \leq a(m_X(\mathfrak{X}), x)$.

Given $(\mathbb{T}, \mathsf{V})$-categories (X, a) and (Y, b), a $(\mathbb{T}, \mathsf{V})$-*functor* $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that $f \cdot a \leq b \cdot Tf$, that is, for each $\mathfrak{x} \in TX$ and $x \in X$, $a(\mathfrak{x}, x) \leq b(Tf(\mathfrak{x}), f(x))$. $(\mathbb{T}, \mathsf{V})$-categories and $(\mathbb{T}, \mathsf{V})$-functors are the objects and morphisms of the category $(\mathbb{T}, \mathsf{V})$-Cat.

Note that each Eilenberg-Moore algebra for $\mathbb{T}$ can be viewed as a $(\mathbb{T}, \mathsf{V})$-category; in fact, we have an embedding

$$\mathsf{Set}^{\mathbb{T}} \hookrightarrow (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}.$$

In particular, for each set X we have the $(\mathbb{T}, \mathsf{V})$-category (TX, m_X) which we denote by $|X|$.

Obviously, each V-category is a $(\mathbb{T}, \mathsf{V})$-category for $\mathbb{T} = \mathbb{1}$ the identity monad "identically" extended to V-Rel. A further class of interesting examples involves the ultrafilter monad $\mathbb{U} = (U, e, m)$. The extension of $U : \mathsf{Set} \to \mathsf{Set}$ to V-Rel of 2.1 can be equivalently described by

$$Ur(\mathfrak{x}, \mathfrak{y}) = \bigwedge_{(A \in \mathfrak{x}, B \in \mathfrak{y})} \bigvee_{(x \in A, y \in B)} r(x, y),$$

for all $r : X \,{\longrightarrow\!\!\!\!\!\mapsto}\, Y$ in V-Rel, $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$. The main result of [1] states that $(\mathbb{U}, 2)$-$\mathsf{Cat} \cong \mathsf{Top}$. In [9] it is shown that $(\mathbb{U}, \mathsf{P}_+)$-$\mathsf{Cat} \cong \mathsf{App}$, the category of approach spaces and non-expansive maps (see [29] for details.)

2.3 The dual $(\mathbb{T}, \mathsf{V})$-category. We have the canonical forgetful functor

$$\begin{aligned} E : (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat} &\to \mathsf{V}\text{-}\mathsf{Cat}, \\ (X, a) &\mapsto (X, a \cdot e_X) \end{aligned}$$

with left adjoint

$$\begin{aligned} E^\circ : \mathsf{V}\text{-}\mathsf{Cat} &\to (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}. \\ (X,a) &\mapsto (X, e_X^\circ \cdot Ta) \end{aligned}$$

Furthermore, the lax extension of T induces an endofunctor

$$\begin{aligned} T : \mathsf{V}\text{-}\mathsf{Cat} &\to \mathsf{V}\text{-}\mathsf{Cat}. \\ (X,a) &\mapsto (TX, Ta) \end{aligned}$$

If m is a natural transformation, we can represent this functor as the composite

$$\begin{tikzcd} & (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \arrow[dr, "M^\circ"] & \\ \mathsf{V}\text{-}\mathsf{Cat} \arrow[ur, "E^\circ"] \arrow[rr, "T"'] & & \mathsf{V}\text{-}\mathsf{Cat}, \end{tikzcd}$$

where $M^\circ : (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \to \mathsf{V}\text{-}\mathsf{Cat}$ is given by $(X,a) \mapsto (TX, Ta \cdot m_X^\circ)$. In fact, given a V-category (X,a), we have

$$T(e_X^\circ \cdot Ta) \cdot m_X^\circ = Te_X^\circ \cdot T^2 a \cdot m_X^\circ = Te_X^\circ \cdot m_X^\circ \cdot Ta = Ta.$$

The functors M° and E° are the key ingredients to define the *dual* $(\mathbb{T},\mathsf{V})$*-category* X^{op} of a $(\mathbb{T},\mathsf{V})$-category $X = (X,a)$: we put $X^{\mathrm{op}} = E^\circ(M^\circ(X)^{\mathrm{op}})$. We remark that X^{op} has as underlying set TX and as structure $a^{\mathrm{op}} := E^\circ((Ta \cdot m_X^\circ)^{\mathrm{op}}) = E^\circ(m_X \cdot Ta^\circ) = e_{TX}^\circ \cdot Tm_X \cdot T^2 a^\circ$. If X is a V-category interpreted as a $(\mathbb{T},\mathsf{V})$-category, i.e. $X = (X, e_X^\circ \cdot Ta)$ for a given V-category structure $a : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$, then

$$X^{\mathrm{op}} = E^\circ(M^\circ(E^\circ(X,a))^{\mathrm{op}}) = E^\circ((TX,Ta)^{\mathrm{op}}),$$

that is, X^{op} is the dual – as a V-category – of $T(X,a)$.

Our Theorem 3.3 shows that this is indeed a reasonable definition.

Finally, for later use we record the following

Lemma. *Let (X,a) be a V-category and (X,α) be a $\mathbb{T}$-algebra. Then $(X, a\cdot\alpha)$ is a $(\mathbb{T},\mathsf{V})$-category if and only if $\alpha : (TX,Ta) \to (X,a)$ is a V-functor.*

Proof. First we remark that from $1_X \le a$ and $1_X = \alpha \cdot e_X$ it follows that $1_X \le (a\cdot\alpha)\cdot e_X$, that is $a\cdot\alpha$ always fulfils the reflexivity axiom. Now, if α is a V-functor, i.e. $\alpha\cdot Ta \le a\cdot\alpha$, then

$$(a\cdot\alpha)\cdot T(a\cdot\alpha) = a\cdot\alpha\cdot Ta\cdot T\alpha \le a\cdot a\cdot\alpha\cdot T\alpha \le (a\cdot\alpha)\cdot m_X.$$

Conversely, if $a\cdot\alpha$ is a $(\mathbb{T},\mathsf{V})$-categorical structure, then

$$\alpha\cdot Ta = \alpha\cdot Ta\cdot T\alpha\cdot Te_X \le a\cdot\alpha\cdot Ta\cdot T\alpha\cdot Te_X \le a\cdot\alpha\cdot m_X\cdot Te_X = a\cdot\alpha. \qquad \square$$

2.4 Kleisli convolution. Many notions and techniques can be transported from V-Cat to $(\mathbb{T},\mathsf{V})$-Cat by formally replacing composition of V-relations by the *Kleisli convolution* (see [22]) defined as

$$b \circ a := b \cdot Ta \cdot m_X^\circ,$$

$$\begin{tikzcd}
TX \arrow[dd, "a", "\shortmid"{marking}] & TY \arrow[dd, "b", "\shortmid"{marking}] & & TX \arrow[r, "m_X^\circ", "\shortmid"{marking}] \arrow[ddr, "b\circ a"', "\shortmid"{marking}] & TTX \arrow[d, "Ta", "\shortmid"{marking}] \\
 & & \longmapsto & & TY \arrow[d, "b", "\shortmid"{marking}] \\
Y & Z & & & Z
\end{tikzcd}$$

for all $a : TX \to\!\!\!\!\!\!\shortmid\;\; Y$ and $b : TY \to\!\!\!\!\!\!\shortmid\;\; Z$ in V-Rel. The V-relation $e_X^\circ : TX \to\!\!\!\!\!\!\shortmid\;\; X$ acts as a lax identity for this convolution, in the following sense:

$$a \circ e_X^\circ = a \qquad \text{and} \qquad e_X^\circ \circ b \geq b,$$

for $a : TX \to\!\!\!\!\!\!\shortmid\;\; Y$ and $b : TY \to\!\!\!\!\!\!\shortmid\;\; X$. Moreover,

$$c \circ (b \circ a) \leq (c \circ b) \circ a$$

if $T : \mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{V}\text{-}\mathsf{Rel}$ preserves composition, and

$$c \circ (b \circ a) \geq (c \circ b) \circ a$$

if $m : TT \to T$ is a natural transformation.

2.5 $(\mathbb{T},\mathsf{V})$-bimodules. Given $(\mathbb{T},\mathsf{V})$-categories (X,a) and (Y,b), a $(\mathbb{T},\mathsf{V})$-*bimodule* (or simply a *bimodule*) $\psi : (X,a) \multimapdotbothvert (Y,b)$ is a V-relation $\psi : TX \to\!\!\!\!\!\!\shortmid\;\; Y$ such that $\psi \circ a \leq \psi$ and $b \circ \psi \leq \psi$ (and therefore $\psi \circ a = \psi$ and $b \circ \psi = \psi$). This means that $\psi \cdot Ta \cdot m_X^\circ \leq \psi$ and $b \cdot T\psi \cdot m_X^\circ \leq \psi$; that is, for $\mathfrak{X} \in T^2X$, $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$ and $y \in Y$,

$$Ta(\mathfrak{X},\mathfrak{x}) \otimes \psi(\mathfrak{x},y) \leq \psi(m_X(\mathfrak{X}),y),$$
$$T\psi(\mathfrak{X},\mathfrak{y}) \otimes b(\mathfrak{y},y) \leq \psi(m_X(\mathfrak{X}),y).$$

Whenever the Kleisli convolution is associative (in particular if $T : \mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{V}\text{-}\mathsf{Rel}$ is a functor and m is a natural transformation: see [22]), bimodules compose. The identities for the composition law are again the $(\mathbb{T},\mathsf{V})$-categorical structures, and we can consider the category $(\mathbb{T},\mathsf{V})$-Mod of $(\mathbb{T},\mathsf{V})$-categories and $(\mathbb{T},\mathsf{V})$-bimodules.

2.6 $(\mathbb{T},\mathsf{V})$-functors as $(\mathbb{T},\mathsf{V})$-bimodules. Analogously to the situation in V-categories, each $(\mathbb{T},\mathsf{V})$-functor $f : (X,a) \to (Y,b)$ defines a pair of bimodules $f_* : (X,a) \multimapdotbothvert (Y,b)$ and $f^* : (Y,b) \multimapdotbothvert (X,a)$ as indicated in the following diagram

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "a"', "\shortmid"{marking}] \arrow[dr, bend left, "f_*", "\shortmid"{marking}] & TY \arrow[d, "b", "\shortmid"{marking}] \arrow[dl, bend left, "f^*", "\shortmid"{marking}] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

that is, $f_* := b \cdot Tf$ and $f^* := f^\circ \cdot b$. In fact, the following assertions are equivalent for $(\mathbb{T},\mathsf{V})$-categories (X,a) and (Y,b) and a function $f : X \to Y$.

(i) $f : (X,a) \to (Y,b)$ is a $(\mathbb{T},\mathsf{V})$-functor.

(ii) $f_* : (X,a) \mathrel{-\!\!\circ\!\!\to} (Y,b)$ is a $(\mathbb{T},\mathsf{V})$-bimodule.

(iii) $f^* : (Y,b) \mathrel{-\!\!\circ\!\!\to} (X,a)$ is a $(\mathbb{T},\mathsf{V})$-bimodule.

Although in general bimodules do not compose, they do compose with functors, as we show next.

Proposition. *Let $f : (X,a) \to (Y,b)$ be a functor, and $\varphi : (Y,b) \mathrel{-\!\!\circ\!\!\to} (Z,c)$ and $\psi : (Z,c) \mathrel{-\!\!\circ\!\!\to} (Y,b)$ be bimodules*

$$\begin{tikzcd} X \arrow[loop left, "a"] \arrow[r, bend left, "f_*"] \arrow[r, "f"] & Y \arrow[l, bend left, "f^*"] \arrow[loop above, "b"] \arrow[r, bend left, "\varphi"] & Z \arrow[l, bend left, "\psi"] \arrow[loop right, "c"] \end{tikzcd}$$

Then:

(1) *$\varphi \circ f_* = \varphi \cdot Tf$ and $f^* \circ \psi = f^\circ \cdot \psi$;*

(2) *$\varphi \circ f_*$ and $f^* \circ \psi$ are bimodules.*

Proof. (1) The latter equality follows from

$$f^* \circ \psi = f^\circ \cdot b \cdot T\psi \cdot m_Z^\circ = f^\circ \cdot \psi,$$

and V-functoriality of f implies

$$\begin{aligned} \varphi \circ f_* = \varphi \circ (b \cdot Tf) &= \varphi \cdot T(b \cdot Tf) \cdot m_X^\circ \\ &\geq \varphi \cdot T(f \cdot a) \cdot m_X^\circ && \text{(by functoriality of } f\text{)} \\ &\geq \varphi \cdot Tf \cdot Ta \cdot m_X^\circ \\ &\geq \varphi \cdot Tf \cdot Ta \cdot Te_X && \text{(since } m_X^\circ \geq Te_X\text{)} \\ &\geq \varphi \cdot Tf, && \text{(since } a \cdot e_X \geq 1_X\text{)} \end{aligned}$$

whereby the property of φ being a bimodule gives us

$$\varphi \circ f_* = \varphi \circ (b \cdot Tf) = \varphi \cdot Tb \cdot T^2 f \cdot m_X^\circ \leq \varphi \cdot Tb \cdot m_Y^\circ \cdot Tf = \varphi \cdot Tf.$$

(2) The bimodule properties of $\varphi \circ f_*$ and $f^* \circ \psi$ follow now from

$$\begin{aligned} &c \circ (\varphi \circ f_*) = c \circ (\varphi \cdot Tf) \leq (c \circ \varphi) \cdot Tf = \varphi \circ f_*, \\ &(\varphi \circ f_*) \circ a = \varphi \cdot Tf \cdot Ta \cdot m_X^\circ \leq \varphi \cdot Tb \cdot T^2 f \cdot m_X^\circ \leq \varphi \cdot Tb \cdot m_Y^\circ \cdot Tf = \varphi \circ f_*, \\ &a \circ (f^* \circ \psi) = a \cdot T(f^\circ \cdot \psi) \cdot m_Z^\circ = a \cdot Tf^\circ \cdot T\psi \cdot m_Z^\circ \leq f^\circ \cdot b \cdot T\psi \cdot m_Z^\circ = f^\circ \cdot \psi = f^* \circ \psi, \\ &(f^* \circ \psi) \circ c = f^\circ \cdot \psi \cdot Tc \cdot m_Z^\circ = f^\circ \cdot (\psi \circ c) = f^\circ \cdot \psi = f^* \circ \psi. \end{aligned}$$

□

Therefore we can define the "whiskering" functors

$$\begin{array}{rcl} (-)\circ f_* : (\mathbb{T},\mathsf{V})\text{-}\mathsf{Mod}(Y,Z) & \longrightarrow & (\mathbb{T},\mathsf{V})\text{-}\mathsf{Mod}(X,Z), \text{ and} \\ \varphi & \longmapsto & \varphi\cdot Tf \end{array}$$

$$\begin{array}{rcl} f^*\circ(-) : (\mathbb{T},\mathsf{V})\text{-}\mathsf{Mod}(Z,Y) & \longrightarrow & (\mathbb{T},\mathsf{V})\text{-}\mathsf{Mod}(Z,X) \\ \psi & \longmapsto & f^\circ\cdot\psi. \end{array}$$

Moreover, given a pair of adjoint bimodules $(\varphi:(Y,b)\,{-\!\!\circ\!\!\to}\,(Z,c))\dashv(\psi:(Z,c)\,{-\!\!\circ\!\!\to}\,(Y,b))$, we have

$$\varphi\circ f_*\dashv f^*\circ\psi,$$

provided that the diagram

$$\begin{tikzcd} T^2X \arrow[r, "m_X"] \arrow[d, "T^2f"'] & TX \arrow[d, "Tf"] \\ T^2Y \arrow[r, "m_Y"] & TY \end{tikzcd}$$

satisfies (BC): $(\varphi\circ f_*)\circ(f^*\circ\psi)\le c$ is always true, since

$$(\varphi\cdot Tf)\circ(f^\circ\cdot\psi)=\varphi\cdot Tf\cdot Tf^\circ\cdot T\psi\cdot m_Z^\circ\le\varphi\cdot T\psi\cdot m_Z^\circ=\varphi\circ\psi\le c,$$

while to conclude that $a\le(f^*\circ\psi)\circ(\varphi\circ f_*)$ we need the (BC) property of the diagram above:

$$a\le f^\circ\cdot b\cdot Tf\le f^\circ\cdot\psi\cdot T\varphi\cdot m_Y^\circ\cdot Tf=f^\circ\cdot\psi\cdot T\varphi\cdot T^2f\cdot m_X^\circ=(f^\circ\cdot\psi)\circ(\varphi\cdot Tf).$$

2.7 Lawvere-complete $(\mathbb{T},\mathsf{V})$-categories.

Definition. A $(\mathbb{T},\mathsf{V})$-category (X,a) is called *Lawvere-complete* if, for each $(\mathbb{T},\mathsf{V})$-category (Y,b) and each pair of adjoint bimodules

$$\begin{tikzcd} Y \arrow[loop left, "b"] \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & X \arrow[l, bend left, "\psi"] \arrow[loop right, "a"] \end{tikzcd}$$

there exists a functor $f:(Y,b)\to(X,a)$ such that $f_*=\varphi$ and $f^*=\psi$.

As for V-categories, such a functor f is said to *represent* the adjunction $\varphi\dashv\psi$. Moreover, analogously to the V-categorical case, Lawvere completeness is fully tested by left adjoint bimodules with domain $(1,p)$, where $p=e_1^\circ$, hence $p(\overset{\bullet}{\star},\star)=k$ and $p(\mathfrak{x},\star)=\bot$ for $\mathfrak{x}\ne\overset{\bullet}{\star}$ in $T1$. In this case we say simply that $f(\star)$ *represents* the adjunction $(\varphi:(Y,b)\to(1,p)\dashv(\psi:(1,p)\to(Y,b))$, whenever it is represented by f.

In case $T1=1$, it is straightforward to check that $f(\star)$ represents $\varphi\dashv\psi$ if and only if, for every $\mathfrak{x}\in TX$, $\psi(\mathfrak{x}):=\psi(\mathfrak{x},\star)=a(\mathfrak{x},f(\star))$.

Proposition. *Assume that either $T1=1$ or that the (*Set*-based) natural transformation m satisfies* (BC)*. Then, for a $(\mathbb{T},\mathsf{V})$-category (X,a), the following conditions are equivalent:*

(i) (X,a) *is Lawvere-complete;*

(ii) *each pair of adjoint bimodules* $\begin{tikzcd}(1,p) \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & (X,a) \arrow[l, bend left, "\psi"]\end{tikzcd}$ *is induced by a functor* $f:(1,p)\to(X,a)$*;*

(iii) *each pair of adjoint bimodules* $\begin{tikzcd}(1,p) \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & (X,a) \arrow[l, bend left, "\psi"]\end{tikzcd}$ *is induced by a map* $f:1\to X$ *(so that* $\varphi = a\cdot Tf$ *and* $\psi = f^\circ\cdot a$*).*

Proof. (iii) $\Rightarrow$ (i): Let $\begin{tikzcd}(Y,b) \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & (X,a) \arrow[l, bend left, "\psi"]\end{tikzcd}$ be a pair of adjoint bimodules. For each $y\in Y$, let $g_y:(1,p)\to(Y,b)$ be the functor that picks y. This functor induces a pair of adjoint bimodules $(g_y)_*\dashv(g_y)^*$, whence we have

$$\begin{tikzcd} 1 \arrow[loop left, "p"] \arrow[r, bend left, "(g_y)_*"] \arrow[r, phantom, "\perp"] & Y \arrow[loop above, "b"] \arrow[l, bend left, "(g_y)^*"] \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & X \arrow[loop right, "a"] \arrow[l, bend left, "\psi"] \end{tikzcd}$$

If m satisfies (BC), we know already that $\varphi_y=\varphi\circ(g_y)_*\dashv\psi_y=(g_y)^*\circ\psi$. In the other case, that is for $T1=1$, it is easy to check that $\varphi_y\dashv\psi_y$. Hence in both cases, by hypothesis, there exists a map $f_y:1\to X$ such that $\varphi_y=b\cdot Tf_y$ and $\psi_y=f_y^\circ\cdot a$. Gluing together the maps $(f_y)_{y\in Y}$ we obtain a map $f:Y\to X$. Then, for $\mathfrak{x}\in TX$ and $y\in Y$,

$$\psi(\mathfrak{x},y)=\psi_y(\mathfrak{x},\star)=f_y^\circ\cdot a(\mathfrak{x},\star)=a(\mathfrak{x},f_y(\star))=a(\mathfrak{x},f(y)),$$

that is, $\psi=f^*=f^\circ\cdot a$. We can show now that f is necessarily a functor:

$$b\cdot Tf^\circ\le b\cdot Tf^\circ\cdot Ta\cdot Te_X\le b\cdot Tf^\circ\cdot Ta\cdot m_X^\circ=b\circ\psi\le\psi=f^\circ\cdot a.$$

This conclues the proof since, by unicity of adjoints, φ is necessarily f_*. $\square$

3 V as a $(\mathbb{T},\mathsf{V})$-category

3.1 The $\mathbb{T}$-algebra structure of V. Our next goal is to explore the notions introduced in the previous section. In particular we are aiming for results which extend known facts about V-categories (as Theorem 1.5 or Theorem 1.6). To do so, from now on *we will always assume that the extension* $T:\mathsf{V}\text{-}\mathsf{Rel}\to\mathsf{V}\text{-}\mathsf{Rel}$ *is constructed as in 2.1 and consequently we assume* V *to be constructively completely distributive. Furthermore, we assume that* $\mathbb{T}=(T,e,m)$ *is non-trivial and that* T *and* m *satisfy (BC).*

Under these conditions, as Manes essentially showed in [31], V can be equipped with a $\mathbb{T}$-algebra structure

$$\begin{aligned}\xi:T\mathsf{V}&\longrightarrow\mathsf{V}\\ \mathfrak{x}&\longmapsto\bigvee\{v\in\mathsf{V}\mid\mathfrak{x}\in T(\uparrow v)\},\end{aligned}$$

where $\uparrow v = \{u \in \mathsf{V} \mid v \leq u\}$.

There is an interesting link between this $\mathbb{T}$-algebra structure and the image under the lax functor $T : \mathsf{V}\text{-}\mathsf{Rel} \to \mathsf{V}\text{-}\mathsf{Rel}$ of the identity $1_{\mathsf{V}} : \mathsf{V} \to \mathsf{V}$ considered as a V-relation $i : 1 \longrightarrow\!\!\!\!\!\!\mapsto \mathsf{V}$, with $i(\star, v) = v$. Let us compute $Ti : T1 \longrightarrow\!\!\!\!\!\!\mapsto T\mathsf{V}$. We consider, for each $v \in \mathsf{V}$, the relation

$$\begin{aligned} i_v : 1 \times \mathsf{V} &\longrightarrow \mathsf{2} \\ (\star, u) &\longmapsto \begin{cases} \text{true} & \text{if } v \leq u, \\ \text{false} & \text{elsewhere,} \end{cases} \end{aligned}$$

hence the diagram

$$\begin{tikzcd} 1 \arrow[rr, "i_v", "|" marking] \arrow[dr, "q_v^\circ"'] & & \mathsf{V}, \\ & \uparrow v \arrow[ur, "p_v"'] & \end{tikzcd}$$

commutes where p_v and q_v are the projections. Now, for each $\mathfrak{x} \in T1$ and $\mathfrak{y} \in T\mathsf{V}$,

$$\begin{aligned} Ti(\mathfrak{x}, \mathfrak{y}) &= \bigvee\{v \in \mathsf{V} \mid T(i_v)(\mathfrak{x}, \mathfrak{y}) = \text{true}\} \\ &= \bigvee\{v \in \mathsf{V} \mid Tp_v \cdot Tq_v^\circ(\mathfrak{x}, \mathfrak{y}) = \text{true}\} \\ &= \bigvee\{v \in \mathsf{V} \mid \exists \mathfrak{z} \in \uparrow v \,:\, Tq_v(\mathfrak{z}) = \mathfrak{x} \text{ and } Tp_v(\mathfrak{z}) = \mathfrak{y}\}, \end{aligned}$$

hence, since T preserves injections and considering Tp_v as an inclusion, we can write

$$Ti(\mathfrak{x}, \mathfrak{y}) = \bigvee\{v \in \mathsf{V} \mid \mathfrak{y} \in T(\uparrow v) \text{ and } Tq_v(\mathfrak{y}) = \mathfrak{x}\} \leq \xi(\mathfrak{y}),$$

by definition of ξ. In particular, if $\mathfrak{x} = Tq(\mathfrak{y})$, for $q : \mathsf{V} \to 1$, then $Ti(\mathfrak{x}, \mathfrak{y}) = \xi(\mathfrak{y})$. Whenever $T1 = 1$, $Tq(\mathfrak{y}) = \dot{\star}$ for every $\mathfrak{y} \in T\mathsf{V}$, whence

$$Ti(\dot{\star}, \mathfrak{y}) = \bigvee\{v \in \mathsf{V} \mid \mathfrak{y} \in T(\uparrow v)\} = \xi(\mathfrak{y}).$$

This link between the extension of T and the $\mathbb{T}$-algebra structure ξ is more general. Whenever necessary, in the sequel we denote the Set-endofunctor T by $T_\circ$, and keep T for its extension to $\mathsf{V}\text{-}\mathsf{Rel}$. When, for a V-relation $r : X \longrightarrow\!\!\!\!\!\!\mapsto Y$, we write $T_\circ r$ we mean $T_\circ$ applied to the map $r : X \times Y \to \mathsf{V}$ which defines the relation r. The interplay between $T_\circ r$ and Tr is stated in the following result.

Proposition. *For any V-relation $r : X \longrightarrow\!\!\!\!\!\!\mapsto Y$, each $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$,*

$$Tr(\mathfrak{x}, \mathfrak{y}) = \bigvee_{\mathfrak{w}\,:\, \substack{T\pi_X(\mathfrak{w}) = \mathfrak{x} \\ T\pi_Y(\mathfrak{w}) = \mathfrak{y}}} \xi \cdot T_\circ r(\mathfrak{w}).$$

Proof. Recall from 2.1 that

$$Tr(\mathfrak{x}, \mathfrak{y}) = \bigvee\{v \in \mathsf{V} \,|\, Tr_v(\mathfrak{x}, \mathfrak{y}) = \text{true}\}.$$

Let $G_v \subseteq X \times Y$ denote the graph of r_v. Directly from the definition of Tr_v we obtain

$$Tr_v(\mathfrak{x}, \mathfrak{y}) = \text{true} \;\Leftrightarrow\; \exists \mathfrak{w} \in TG_v \,:\, T\pi_X(\mathfrak{w}) = \mathfrak{x} \;\&\; T\pi_Y(\mathfrak{w}) = \mathfrak{y}.$$

Now note that, for $\uparrow v := \{u \in \mathsf{V} \mid v \le u\}$,

$$\begin{tikzcd} G_v \arrow[r] \arrow[d] & \uparrow v \arrow[d] \\ X \times Y \arrow[r, "r"] & \mathsf{V} \end{tikzcd}$$

is a pullback. Hence, since $T_\circ$ has (BC), for $\mathfrak{w} \in T(X \times Y)$ we have

$$\mathfrak{w} \in TG_v \;\Leftrightarrow\; T_\circ r(\mathfrak{w}) \in \uparrow v.$$

Therefore, using also (ccd) of V, we conclude

$$\bigvee\{v \in \mathsf{V} \mid \exists \mathfrak{w} \in TG_v\; T\pi_X(\mathfrak{w}) = \mathfrak{x} \;\&\; T\pi_Y(\mathfrak{w}) = \mathfrak{y}\} = \bigvee_{\mathfrak{w}:\; \substack{T\pi_X(\mathfrak{w})=\mathfrak{x} \\ T\pi_Y(\mathfrak{w})=\mathfrak{y}}} \xi \cdot T_\circ r(\mathfrak{w}).$$

□

Remark. Besides being the structure map of an Eilenberg-Moore algebra, $\xi : T\mathsf{V} \to \mathsf{V}$ satisfies also the inequalities

$$\otimes \cdot \langle \xi \cdot T_\circ \pi_1, \xi \cdot T_\circ \pi_2 \rangle \le \xi \cdot T_\circ(\otimes) \qquad \text{and} \qquad k \cdot ! \le \xi \cdot T_\circ k.$$

$$\begin{tikzcd} T(\mathsf{V} \times \mathsf{V}) \arrow[r, "T_\circ(\otimes)"] \arrow[d, "\langle \xi \cdot T_\circ \pi_1, \xi \cdot T_\circ \pi_2 \rangle"'] \arrow[dr, phantom, "\le"] & T\mathsf{V} \arrow[d, "\xi"] \\ \mathsf{V} \times \mathsf{V} \arrow[r, "\otimes"'] & \mathsf{V} \end{tikzcd} \qquad \begin{tikzcd} T1 \arrow[r, "T_\circ k"] \arrow[d, "!"'] \arrow[dr, phantom, "\le"] & T\mathsf{V} \arrow[d, "\xi"] \\ 1 \arrow[r, "k"'] & \mathsf{V} \end{tikzcd}$$

Recall that we assume $Tf = T_\circ f$ for each Set-map $f : X \to Y$; this condition requires and implies equality in the latter inequality (see [24]).

3.2 The canonical $(\mathbb{T}, \mathsf{V})$-categorical structure of V. The composition of the natural V-categorical and $\mathbb{T}$-algebra structures of V defines an interesting structure, $\hom_\xi$, of a $(\mathbb{T}, \mathsf{V})$-category on V

$$T\mathsf{V} \xrightarrow{\hom_\xi} \mathsf{V} = (T\mathsf{V} \xrightarrow{\xi} \mathsf{V} \xrightarrow{\hom} \mathsf{V}),$$

as we show next.

Proposition. *$\xi : (T\mathsf{V}, T\hom) \to (\mathsf{V}, \hom)$ is a V-functor.*

Proof. We have to show that $\xi \cdot T\hom \le \hom \cdot \xi$, or, equivalently, $T\hom \le \xi^\circ \cdot \hom \cdot \xi$. This means that, for $\mathfrak{x}, \mathfrak{y} \in T\mathsf{V}$,

$$T\hom(\mathfrak{x}, \mathfrak{y}) \le \hom(\xi(\mathfrak{x}), \xi(\mathfrak{y})).$$

We consider again the V-relation $i : 1 \to\!\!\!\!\!\!+\;\, \mathsf{V}$, and compute $1 \xrightarrow{i} \mathsf{V} \xrightarrow{\hom} \mathsf{V}$:

$$\hom \cdot i(\star, v) = \bigvee_{u \in \mathsf{V}} i(\star, u) \otimes \hom(u, v) = \bigvee_{u \in \mathsf{V}} u \otimes \hom(u, v) \le v;$$

that is $\hom \cdot i \le i$. Hence $T\hom \cdot Ti \le T(\hom \cdot i) \le Ti$, and so, for $\mathfrak{x}, \mathfrak{y} \in T\mathsf{V}$ and $\mathfrak{z} = Tq(\mathfrak{x})$ as in Section 3.1, we have

$$\xi(\mathfrak{x}) \otimes T\hom(\mathfrak{x}, \mathfrak{y}) = Ti(\mathfrak{z}, \mathfrak{x}) \otimes T\hom(\mathfrak{x}, \mathfrak{y}) \le Ti(\mathfrak{z}, \mathfrak{y}) \le \xi(\mathfrak{y}),$$

and therefore

$$T\hom(\mathfrak{x},\mathfrak{y}) \leq \hom(\xi(\mathfrak{x}),\xi(\mathfrak{y}))$$

as claimed. $\square$

Corollary. $(\mathsf{V},\hom_\xi)$ *is a* $(\mathbb{T},\mathsf{V})$*-category.*

Proof. Follows from the proposition above and Lemma 2.3. $\square$

3.3 The tensor product. The tensor product in V defines in a natural way a (not necessarily closed) tensor product structure in $(\mathbb{T},\mathsf{V})$-Cat. Given $(\mathbb{T},\mathsf{V})$-categories $X=(X,a)$ and $Y=(Y,b)$, we put $X\otimes Y=(X\times Y,a\otimes b)$ where $a\otimes b(\mathfrak{w},(x,y))=a(T\pi_X(\mathfrak{w}),x)\otimes b(T\pi_Y(\mathfrak{w}),y)$ for all $\mathfrak{w}\in T(X\times Y)$, $x\in X$ and $y\in Y$. One easily verifies reflexivity of $a\otimes b$, while transitivity holds if and only if $\otimes\cdot\langle\xi\cdot T_{\mathrm{o}}\pi_1,\xi\cdot T_{\mathrm{o}}\pi_2\rangle=\xi\cdot T_{\mathrm{o}}(\otimes)$ (see Remark 3.1 and [24]) *which we assume from now on.* We remark that this condition guarantees that $\mathbb{T}$ is a *lax Hopf monad* on V-Rel (see [32]) where the tensor product in V is naturally extended to V-Rel. However, we will not develop this aspect here.

It is well-known that in general the functor $X\otimes(-):(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ has no right adjoint as, for example, Top is not Cartesian closed. The problem of characterising those $(\mathbb{T},\mathsf{V})$-categories $X=(X,a)$ such that tensoring with X has a right adjoint is studied in [24].

Theorem. *Let m be a natural transformation. For $(\mathbb{T},\mathsf{V})$-categories (X,a) and (Y,b) and a V-relation $\psi:TX\relbar\!\!\!\rightarrow Y$, the following assertions are equivalent.*

(i) $\psi:(X,a)\relbar\!\!\circ\!\!\rightarrow(Y,b)$ *is a* $(\mathbb{T},\mathsf{V})$*-bimodule.*

(ii) *Both* $\psi:|X|\otimes Y\to\mathsf{V}$ *and* $\psi:X^{\mathrm{op}}\otimes Y\to\mathsf{V}$ *are* $(\mathbb{T},\mathsf{V})$*-functors.*

Proof. Assume that $\psi:(X,a)\relbar\!\!\circ\!\!\rightarrow(Y,b)$ is a $(\mathbb{T},\mathsf{V})$-bimodule. First observe that, for $\mathfrak{W}\in T(TX\times Y)$,

$$\xi\cdot T_{\mathrm{o}}\psi(\mathfrak{W})\leq T\psi(T_{\mathrm{o}}\pi_{TX}(\mathfrak{W}),T_{\mathrm{o}}\pi_Y(\mathfrak{W})).$$

Let $\mathfrak{W}\in T(TX\times Y)$, $\mathfrak{x}\in TX$ and $y\in Y$. To see that $\psi:|X|\otimes Y\to\mathsf{V}$ is a $(\mathbb{T},\mathsf{V})$-functor, note that the structure c on $|X|\otimes Y$ is given by

$$c(\mathfrak{W},(\mathfrak{x},y))=\begin{cases}\bot & \text{if } \mathfrak{x}\neq m_X(T_{\mathrm{o}}\pi_{TX}(\mathfrak{W})),\\ b(T\pi_Y(\mathfrak{W}),y) & \text{if } \mathfrak{x}=m_X(T_{\mathrm{o}}\pi_{TX}(\mathfrak{W})).\end{cases}$$

Assume $\mathfrak{x}=m_X(T_{\mathrm{o}}\pi_{TX}(\mathfrak{W}))$. Since

$$b(T_{\mathrm{o}}\pi_Y(\mathfrak{W}),y)\leq\hom(\xi\cdot T_{\mathrm{o}}\psi(\mathfrak{W}),\psi(\mathfrak{x},y))$$

is equivalent to

$$\xi\cdot T_{\mathrm{o}}\psi(\mathfrak{W})\otimes b(T_{\mathrm{o}}\pi_Y(\mathfrak{W}),y)\leq\psi(\mathfrak{x},y),$$

it follows at once that $\psi:|X|\otimes Y\to\mathsf{V}$ is a $(\mathbb{T},\mathsf{V})$-functor. We show now that $\psi:X^{\mathrm{op}}\otimes Y\to\mathsf{V}$ is a $(\mathbb{T},\mathsf{V})$-functor. As above we have that (with $a^{\mathrm{op}}=e_{TX}^\circ\cdot Tm_X\cdot T^2a^\circ$ the structure on X^{op})

$$a^{\mathrm{op}}(T_{\mathrm{o}}\pi_{TX}(\mathfrak{W}),\mathfrak{x})\otimes b(T_{\mathrm{o}}\pi_Y(\mathfrak{W}),y)\leq\hom(\xi\cdot T_o\psi(\mathfrak{W}),\psi(\mathfrak{x},y))$$

is equivalent to

$$\xi \cdot T_\circ\psi(\mathfrak{W}) \otimes a^{\mathrm{op}}(T_\circ\pi_{TX}(\mathfrak{W}), \mathfrak{x}) \otimes b(T_\circ\pi_Y(\mathfrak{W}), y) \leq \psi(\mathfrak{x}, y).$$

Now

$$\begin{aligned}
\xi \cdot T_\circ\psi(\mathfrak{W}) &\otimes a^{\mathrm{op}}(T_\circ\pi_{TX}(\mathfrak{W}), \mathfrak{x}) \otimes b(T_\circ\pi_Y(\mathfrak{W}), y) \\
&\leq T^2a \cdot Tm_X^\circ \cdot e_{TX}(\mathfrak{x}, T_\circ\pi_{TX}(\mathfrak{W})) \otimes T\psi(T_\circ\pi_{TX}(\mathfrak{W}), T_\circ\pi_Y(\mathfrak{W})) \otimes b(T_\circ\pi_Y(\mathfrak{W}), y) \\
&\leq b \cdot T\psi \cdot T^2a \cdot Tm_X^\circ \cdot m_X^\circ(\mathfrak{x}, y) \\
&\leq b \cdot T\psi \cdot m_X^\circ \cdot Ta \cdot m_X^\circ(\mathfrak{x}, y) \\
&= \psi \cdot Ta \cdot m_X^\circ(\mathfrak{x}, y) = \psi(\mathfrak{x}, y).
\end{aligned}$$

Now assume that $\psi : |X| \otimes Y \to \mathsf{V}$ and $\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$ are $(\mathbb{T}, \mathsf{V})$-functors. Functoriality of $\psi : |X| \otimes Y \to \mathsf{V}$ implies, for all $\mathfrak{x} \in TX$ and $y \in Y$,

$$\begin{aligned}
\psi(\mathfrak{x}, y) &\geq \bigvee_{\substack{\mathfrak{X} \in TTX: \\ m_X(\mathfrak{X}) = \mathfrak{x}; \\ \mathfrak{y} \in TY}} \bigvee \left\{ \xi \cdot T_\circ\psi(\mathfrak{W}) \otimes b(\mathfrak{y}, y) \;\middle|\; \mathfrak{W} \in T(TX \times Y) : T\pi_{TX}(\mathfrak{W}) = \mathfrak{X}, T\pi_Y(\mathfrak{W}) = \mathfrak{y} \right\} \\
&= \bigvee_{\substack{\mathfrak{X} \in TTX: \\ m_X(\mathfrak{X}) = \mathfrak{x}; \\ \mathfrak{y} \in TY}} T\psi(\mathfrak{X}, \mathfrak{y}) \otimes b(\mathfrak{y}, y) \\
&= \bigvee_{\substack{\mathfrak{X} \in TTX: \\ m_X(\mathfrak{X}) = \mathfrak{x}}} b \cdot T\psi(\mathfrak{X}, y) \\
&= b \cdot T\psi \cdot m_X^\circ(\mathfrak{x}, y).
\end{aligned}$$

On the other hand, by functoriality of $\psi : X^{\mathrm{op}} \otimes Y \to \mathsf{V}$, for all $\mathfrak{x} \in TX$ and $y \in Y$ we have

$$\begin{aligned}
\psi(\mathfrak{x}, y) &\geq \bigvee_{\substack{\mathfrak{X} \in TTX, \\ \mathfrak{y} \in TY}} \bigvee \left\{ \xi \cdot T_\circ\psi(\mathfrak{W}) \otimes b(\mathfrak{y}, y) \otimes a^{\mathrm{op}}(\mathfrak{X}, \mathfrak{x}) \;\middle|\; \mathfrak{W} \in T(TX \times Y) : T\pi_{TX}(\mathfrak{W}) = \mathfrak{X}, T\pi_Y(\mathfrak{W}) = \mathfrak{y} \right\} \\
&= \bigvee_{\substack{\mathfrak{X} \in TTX, \\ \mathfrak{y} \in TY}} T\psi(\mathfrak{X}, \mathfrak{y}) \otimes b(\mathfrak{y}, y) \otimes T^2a \cdot Tm_X^\circ \cdot e_{TX}(\mathfrak{x}, \mathfrak{X}) \\
&= b \cdot T\psi \cdot T^2a \cdot Tm_X^\circ \cdot e_{TX}(\mathfrak{x}, \mathfrak{X}) \\
&\geq b \cdot e_Y \cdot \psi \cdot Ta \cdot m_X^\circ(\mathfrak{x}, y) \\
&\geq \psi \cdot Ta \cdot m_X^\circ(\mathfrak{x}, y).
\end{aligned}$$

□

3.4 V is Lawvere-complete. The proof that $(\mathsf{V}, \hom_\xi)$ is a Lawvere complete $(\mathbb{T}, \mathsf{V})$-category is based on the corresponding proof for V-categories. However, here we need to consider further assumptions. Our next result analyses the main assumption of our subsequent Theorem. (For an alternative proof of this result see [26].)

Lemma. *Assume that $T1 = 1$. Then $T(\hom_\xi) \cdot m_{\mathsf{V}}^\circ = \xi^\circ \cdot \hom \cdot \xi$, i.e., for every $\mathfrak{v}, \mathfrak{w} \in T\mathsf{V}$, $a(\mathfrak{v}, \mathfrak{w}) = \hom(\xi(\mathfrak{v}), \xi(\mathfrak{w}))$, provided that*

$$\begin{tikzcd}
T\mathsf{V} \arrow[r, "T_\circ(\hom(u,-))"] \arrow[d, "\xi"'] \arrow[dr, phantom, "\leq"] & T\mathsf{V} \arrow[d, "\xi"] \\
\mathsf{V} \arrow[r, "\hom(u,-)"'] & \mathsf{V}
\end{tikzcd}$$

for each $u \in \mathsf{V}$*. This inequality holds if* $\hom(u,-)$ *preserves non-empty suprema.*

Proof. First observe that

$$\begin{aligned} T(\hom_\xi)\cdot m_{\mathsf{V}}^\circ &= T\hom\cdot T\xi\cdot m_{\mathsf{V}}^\circ \\ &\le \xi^\circ\cdot\hom\cdot\xi\cdot T\xi\cdot m_{\mathsf{V}}^\circ \qquad \text{(because } \xi \text{ is a } \mathsf{V}\text{-functor, by Proposition 3.2)} \\ &= \xi^\circ\cdot\hom\cdot\xi\cdot m_{\mathsf{V}}\cdot m_{\mathsf{V}}^\circ \\ &\le \xi^\circ\cdot\hom\cdot\xi. \end{aligned}$$

On the other hand, for $\mathfrak{u},\mathfrak{v}\in T\mathsf{V}$, we have

$$\begin{aligned} T(\hom_\xi)\cdot m_{\mathsf{V}}^\circ(\mathfrak{u},\mathfrak{v}) &\ge T\hom_\xi(\overset{\bullet}{\mathfrak{u}},\mathfrak{v}) \\ &= T\hom(T_{\mathrm{o}}\xi(\overset{\bullet}{\mathfrak{u}}),\mathfrak{v}) \\ &= T\hom(\overset{\bullet}{\xi(\mathfrak{u})},\mathfrak{v}) \\ &= \xi\cdot T_{\mathrm{o}}\hom\cdot T_{\mathrm{o}}\langle\xi(\mathfrak{u}),1_{\mathsf{V}}\rangle(\mathfrak{v}) \qquad (*) \\ &\ge \hom(\xi(\mathfrak{u}),-)\cdot\xi(\mathfrak{v}) = \hom(\xi(\mathfrak{u}),\xi(\mathfrak{v})). \end{aligned}$$

To see (*), just observe that $T_{\mathrm{o}}\langle\xi(\mathfrak{u}),1_{\mathsf{V}}\rangle(\mathfrak{v})$ is the only element of $T(\mathsf{V}\times\mathsf{V})$ which projects to both $\overset{\bullet}{\xi(\mathfrak{u})}$ and $\mathfrak{v}$.

Assume now that $\hom(u,-)$ preserves non-empty suprema and let $u\in\mathsf{V}$ and $\mathfrak{u}\in T\mathsf{V}$. Then

$$\begin{aligned} \hom(u,\xi(\mathfrak{u})) &= \hom(u,\bigvee\{v\in\mathsf{V}\mid \mathfrak{u}\in T(\uparrow v)\}) \\ &= \bigvee\{\hom(u,v)\mid v\in\mathsf{V},\mathfrak{u}\in T(\uparrow v)\} \\ &\le \bigvee\{v\in\mathsf{V}\mid T_{\mathrm{o}}\hom(u,-)(\mathfrak{u})\in T(\uparrow v)\}. \end{aligned}$$

□

Theorem. *Assume that* $T1=1$*. Then* $(\mathsf{V},\hom_\xi)$ *is a Lawvere-complete* $(\mathbb{T},\mathsf{V})$*-category provided that* $T(\hom_\xi)\cdot m_{\mathsf{V}}^\circ=\xi^\circ\cdot\hom\cdot\xi$.

Proof. Let

$$\begin{tikzcd} 1 \arrow[loop left, "p"] \arrow[r, bend left, "\varphi"] \arrow[r, phantom, "\perp"] & \mathsf{V} \arrow[l, bend left, "\psi"] \arrow[loop right, "\hom_\xi"] \end{tikzcd}$$

be a pair of adjoint bimodules. By the previous theorem we know that:

$$\text{(1)} \qquad \begin{aligned} \varphi \text{ bimodule} &\iff \varphi:(\mathsf{V},\hom_\xi)\to(\mathsf{V},\hom_\xi) \text{ is a } (\mathbb{T},\mathsf{V})\text{-functor} \\ &\iff \forall\mathfrak{v}\in T\mathsf{V}\;\;\forall v\in\mathsf{V}\;\;\hom(\xi(\mathfrak{v}),v)\le\hom(\xi\cdot T\varphi(\mathfrak{v}),\varphi(v)). \end{aligned}$$

In particular, for every $\mathfrak{v}\in T\mathsf{V}$, $k\le\hom(\xi(\mathfrak{v}),\xi(\mathfrak{v}))\le\hom(\xi\cdot T\varphi(\mathfrak{v}),\varphi\cdot\xi(\mathfrak{v}))$, hence $\xi\cdot T\varphi(\mathfrak{v})\le\varphi\cdot\xi(\mathfrak{v})$.

Let $a=M^\circ(\hom_\xi)$, which, by hypothesis, is $\xi^\circ\cdot\hom\cdot\xi$. Then:

$$\text{(2)} \qquad \begin{aligned} \psi \text{ bimodule} &\iff \psi:(T\mathsf{V},a^\circ)\to(\mathsf{V},\hom) \text{ is a } \mathsf{V}\text{-functor} \\ &\iff \forall\mathfrak{v},\mathfrak{w}\in T\mathsf{V}\;\;a(\mathfrak{v},\mathfrak{w})\le\hom(\psi(\mathfrak{w}),\psi(\mathfrak{v})). \end{aligned}$$

Finally,

$$(3) \qquad \varphi \dashv \psi \iff \begin{cases} (a) & \varphi\circ\psi \le \hom\cdot\xi \iff \forall\mathfrak{v}\in T\mathsf{V}\,\forall v\in\mathsf{V}\ \psi(\mathfrak{v})\otimes\varphi(v)\le\hom(\xi(\mathfrak{v}),v),\\ (b) & p\le\psi\circ\varphi \iff k\le\displaystyle\bigvee_{\mathfrak{u}\in T\mathsf{V}}\psi(\mathfrak{u})\otimes\xi(T_\circ\varphi(\mathfrak{u})).\end{cases}$$

We will show that the adjunction $\varphi\dashv\psi$ is represented by $\psi(\dot{k})$, where $\dot{k}=e_{\mathsf{V}}(k)$. That is, according to 2.7,

$$\forall v\in T\mathsf{V}\ \ \psi(\mathfrak{v})=\hom_\xi(\mathfrak{v},\psi(\dot{k})).$$

Similarly to the proof of Theorem 1.6, we split our argument in three steps:

(1st) $\psi(\dot{k})=\displaystyle\bigvee_{\mathfrak{v}\in T\mathsf{V}}\psi(\mathfrak{v})\otimes\xi(\mathfrak{v})$:

"$\le$" is immediate; for "$\ge$" we argue as follows:

$$\begin{aligned}\psi(\mathfrak{v})\otimes\xi(\mathfrak{v}) &= \psi(\mathfrak{v})\otimes\hom(\xi(\dot{k}),\xi(\mathfrak{v}))\\ &= \psi(\mathfrak{v})\otimes a(\dot{k},\mathfrak{v}) && \text{(by hypothesis)}\\ &\le \psi(\mathfrak{v})\otimes\hom(\psi(\mathfrak{v}),\psi(\dot{k})) && \text{(by (2))}\\ &\le \psi(\dot{k}).\end{aligned}$$

(2nd) $\forall\mathfrak{v}\in T\mathsf{V}\ \hom_\xi(\mathfrak{v},\psi(\dot{k}))=\displaystyle\bigvee_{\mathfrak{u}\in T\mathsf{V}}\hom(\xi(\mathfrak{v}),\xi(\mathfrak{u}))\otimes\psi(\mathfrak{u})$:

To check "$\ge$" we just observe that

$$\xi(\mathfrak{v})\otimes(\hom(\xi(\mathfrak{v}),\xi(\mathfrak{u}))\otimes\psi(\mathfrak{u}))\le\xi(\mathfrak{u})\otimes\psi(\mathfrak{u})\le\psi(\dot{k}),$$

by our **(1st)** equality.

For "$\le$", first note that

$$\psi(\dot{k})\otimes\varphi(\xi(\mathfrak{u}))\le\hom(\xi(\dot{k}),\xi(\mathfrak{u}))=\hom(k,\xi(\mathfrak{u}))=\xi(\mathfrak{u})$$

from which follows

$$(4) \qquad \xi(T\varphi(\mathfrak{u}))\le\varphi(\xi(\mathfrak{u}))\le\hom(\psi(\dot{k}),\xi(\mathfrak{u})).$$

From that we conclude that

$$\begin{aligned}\hom(\xi(\mathfrak{v}),\psi(\dot{k})) &\le \hom(\xi(\mathfrak{v}),\psi(\dot{k}))\otimes\bigvee_{\mathfrak{u}\in T\mathsf{V}}\psi(\mathfrak{u})\otimes\xi(T\varphi(\mathfrak{u})) && \text{(by (3b))}\\ &= \bigvee_{\mathfrak{u}\in T\mathsf{V}}\hom(\xi(\mathfrak{v}),\psi(\dot{k}))\otimes\psi(\mathfrak{u})\otimes\xi(T\varphi(\mathfrak{u}))\\ &\le \bigvee_{\mathfrak{u}\in T\mathsf{V}}\hom(\xi(\mathfrak{v}),\psi(\dot{k}))\otimes\hom(\psi(\dot{k}),\xi(\mathfrak{u}))\otimes\psi(\mathfrak{u}) && \text{(by (4))}\\ &\le \bigvee_{\mathfrak{u}\in T\mathsf{V}}\hom(\xi(\mathfrak{v}),\xi(\mathfrak{u}))\otimes\psi(\mathfrak{u}).\end{aligned}$$

(3rd) $\forall\mathfrak{v}\in T\mathsf{V}\ \psi(\mathfrak{v})=\displaystyle\bigvee_{\mathfrak{u}\in T\mathsf{V}}a(\mathfrak{v},\mathfrak{u})\otimes\psi(\mathfrak{u})$:

For "$\le$" take $\mathfrak{u}=\mathfrak{v}$; for "$\ge$" we use (2): $a^\circ(\mathfrak{u},\mathfrak{v})\otimes\psi(\mathfrak{u})\le\hom(\psi(\mathfrak{u}),\psi(\mathfrak{v}))\otimes\psi(\mathfrak{u})\le\psi(\mathfrak{v})$.

Finally, using the hypothesis that, for every $\mathfrak{u},\mathfrak{v}\in T\mathsf{V}$, $a(\mathfrak{v},\mathfrak{u})=\hom(\xi(\mathfrak{v}),\xi(\mathfrak{u}))$, one can conclude that $\psi(\dot{k})$ represents $\varphi\dashv\psi$, since from **(2nd)** and **(3rd)** it follows that

$$\forall\mathfrak{v}\in T\mathsf{V}\ \ \psi(\mathfrak{v})=\hom_\xi(\mathfrak{v},\psi(\dot{k})).$$ □

4 A Yoneda Lemma for $(\mathbb{T},\mathsf{V})$-categories

4.1 Function spaces. In this section we wish to obtain the analogue result to Corollary 1.5 in the setting of $(\mathbb{T},\mathsf{V})$-categories. This in turn requires an understanding of the right adjoint to $X\otimes(-):(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$, a problem studied in [24]. From there we import the following result.

Proposition. *Let $X=(X,a)$ be a $(\mathbb{T},\mathsf{V})$-category. Then $X\otimes(-):(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ has a right adjoint $(-)^X$ provided that $a\cdot Ta=a\cdot m_X$.*

Certainly, each (Eilenberg-Moore) $\mathbb{T}$-algebra, considered as a $(\mathbb{T},\mathsf{V})$-category, satisfies the condition above. Moreover, the $(\mathbb{T},\mathsf{V})$-categorical structure (X,a) induced by any V-category $X=(X,r)$ (see 2.3) satisfies this condition if $Te_X\cdot e_X=m_X^\circ\cdot e_X$.

Let $X=(X,a)$ and (Y,b) be $(\mathbb{T},\mathsf{V})$-categories, and assume that $a\cdot Ta=a\cdot m_X$. Then Y^X has as underlying set

$$\{h:(X,a)\otimes(1,p)\to(Y,b)\mid h\text{ is a }(\mathbb{T},\mathsf{V})\text{-functor}\},$$

thanks to the bijection (with $P=(1,p)$)

$$\frac{X\otimes P\to Y}{P\to Y^X}.$$

The structure $[\![a,b]\!]$ on Y^X is the largest structure making the evaluation map

$$\mathrm{ev}:X\otimes Y^X\to Y,\ (x,h)\mapsto h(x)$$

a $(\mathbb{T},\mathsf{V})$-functor: for $\mathfrak{p}\in T(Y^X)$ and $h\in Y^X$ we have

$$[\![a,b]\!](\mathfrak{p},h)=\bigvee\left\{v\in\mathsf{V}\;\middle|\;\forall\mathfrak{q}\in T\pi_{Y^X}^{-1}(\mathfrak{p}),x\in X\ \ a(T\pi_X(\mathfrak{q}),x)\otimes v\le b(T\mathrm{ev}(\mathfrak{q}),h(x))\right\}.$$

4.2 The Yoneda Embedding. By Theorem 3.3, the bimodule $a:X\mathrel{-\!\!\!\circ\!\!\!\rightarrow}X$ gives rise to $(\mathbb{T},\mathsf{V})$-functors $a:|X|\otimes X\to\mathsf{V}$ and $a:X^{\mathrm{op}}\otimes X\to\mathsf{V}$. According to the considerations above, we obtain a $(\mathbb{T},\mathsf{V})$-functor $y=\ulcorner a\urcorner:X\to\mathsf{V}^{|X|}$. Our next result should be compared with Corollary 1.5.

Theorem (Yoneda)**.** *Let $X=(X,a)$ be a $(\mathbb{T},\mathsf{V})$-category. Then the following assertions hold.*

(a) For all $\mathfrak{x}\in TX$ and $\varphi\in\mathsf{V}^{|X|}$, $[\![m_X,\hom_\xi]\!](Ty(\mathfrak{x}),\varphi)\le\varphi(\mathfrak{x})$.

(b) Let $\varphi\in\mathsf{V}^{|X|}$. Then

$$\forall\mathfrak{x}\in TX\ \varphi(\mathfrak{x})\le[\![m_X,\hom_\xi]\!](Ty(\mathfrak{x}),\varphi)\qquad\iff\qquad\varphi:X^{\mathrm{op}}\to\mathsf{V}\textit{ is a }(\mathbb{T},\mathsf{V})\textit{-functor.}$$

Proof. Note that the diagrams

$$\begin{tikzcd} & \mathsf{V} \\ TX\times X \arrow[ur, "a"] \arrow[r, "1_{TX}\times y"'] & TX\times\mathsf{V}^{|X|} \arrow[u, "\mathrm{ev}"'] \end{tikzcd}\qquad\qquad\begin{tikzcd} TX\times X \arrow[r, "1_{TX}\times y"] \arrow[d, "\pi_2"'] & TX\times\mathsf{V}^{|X|} \arrow[d, "\pi_2"] \\ X \arrow[r, "y"'] & \mathsf{V}^{|X|} \end{tikzcd}$$

commute, where the right-hand side diagram is even a pullback. Let $\mathfrak{x} \in TX$ and $\varphi \in \mathsf{V}^{|X|}$. Hence

$$\begin{aligned}
&[\![m_X, \hom_\xi]\!](T_{\mathrm{o}}y(\mathfrak{x}), \varphi) \\
&\qquad = \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y}), \mathfrak{W} \in T(TX \times \mathsf{V}^{|X|}) \\
&\qquad\qquad (T_{\mathrm{o}}\pi_1(\mathfrak{W}) = \mathfrak{Y} \,\&\, T_{\mathrm{o}}\pi_2(\mathfrak{W}) = T_{\mathrm{o}}y(\mathfrak{x})) \Rightarrow v \le \hom(\xi \cdot T_{\mathrm{o}}\,\mathrm{ev}(\mathfrak{W}), \varphi(\mathfrak{y}))\} \\
&\qquad = \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y}), \mathfrak{W} \in T(TX \times X) \\
&\qquad\qquad (T_{\mathrm{o}}\pi_1(\mathfrak{W}) = \mathfrak{Y} \,\&\, T_{\mathrm{o}}\pi_2(\mathfrak{W}) = \mathfrak{x}) \Rightarrow v \le \hom(\xi \cdot T_{\mathrm{o}}a(\mathfrak{W}), \varphi(\mathfrak{y}))\} \\
&\qquad = \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y}) \;\; v \le \bigwedge_{\substack{\mathfrak{W} \in T(TX \times X) \\ T_{\mathrm{o}}\pi_1(\mathfrak{W}) = \mathfrak{Y} \\ T_{\mathrm{o}}\pi_2(\mathfrak{W}) = \mathfrak{x}}} \hom(\xi \cdot T_{\mathrm{o}}a(\mathfrak{W}), \varphi(\mathfrak{y}))\} \\
&\qquad = \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y}) \;\; v \le \hom(\bigvee_{\substack{\mathfrak{W} \in T(TX \times X) \\ T_{\mathrm{o}}\pi_1(\mathfrak{W}) = \mathfrak{Y} \\ T_{\mathrm{o}}\pi_2(\mathfrak{W}) = \mathfrak{x}}} \xi \cdot T_{\mathrm{o}}a(\mathfrak{W}), \varphi(\mathfrak{y}))\} \\
&\qquad = \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in m_X^{-1}(\mathfrak{y}) \;\; v \le \hom(Ta(\mathfrak{Y}, \mathfrak{x}), \varphi(\mathfrak{y}))\} \\
&\qquad = \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX \;\; Ta \cdot m_X^\circ(\mathfrak{y}, \mathfrak{x}) \otimes v \le \varphi(\mathfrak{y})\}.
\end{aligned}$$

In particular we have

$$v = k \otimes v \le Ta \cdot m_X^\circ(\mathfrak{x}, \mathfrak{x}) \otimes v \le \varphi(\mathfrak{x}),$$

which proves (a). On the other hand, $\varphi : (TX, Ta \cdot m_X^\circ) \to (\mathsf{V}, \hom)$ is a V-functor if and only if

$$Ta \cdot m_X^\circ(\mathfrak{y}, \mathfrak{x}) \otimes \varphi(\mathfrak{x}) \le \varphi(\mathfrak{y})$$

for all $\mathfrak{y}, \mathfrak{x} \in TX$, from which (b) follows. □

We put $\hat{X} = (\hat{X}, \hat{a})$ where $\hat{X} := \{\varphi \in \mathsf{V}^{|X|} \mid \varphi : X^{\mathrm{op}} \to \mathsf{V}$ is a $(\mathbb{T}, \mathsf{V})$-functor$\}$ considered as a subcategory of $\mathsf{V}^{|X|}$. Recall that $a : X^{\mathrm{op}} \otimes X \to \mathsf{V}$ is a $(\mathbb{T}, \mathsf{V})$-functor, and therefore $a(-, x) : X^{\mathrm{op}} \otimes P \to \mathsf{V}$ is a $(\mathbb{T}, \mathsf{V})$-functor for each $x \in X$. If $T1 = 1$, then $P = (1, p) = (1, k)$ is the neutral element for $\otimes$ and we can restrict the Yoneda functor y to $\hat{X}$.

Corollary. *Assume $T1 = 1$. Then the Yoneda functor $y : X \to \hat{X}$ is full and faithful.*

If $Te_X \cdot e_X = m_X^\circ \cdot e_X$, we also might consider the transpose $y_0 = \ulcorner a \urcorner : X \to \mathsf{V}^{X^{\mathrm{op}}}$ of $a : X^{\mathrm{op}} \otimes X \to \mathsf{V}$ as below. However, unlike the situation for V-categories, in general we do not have $\hat{X} \cong \mathsf{V}^{X^{\mathrm{op}}}$ (see example below).

Proposition (Yoneda II)**.** *Assume that $Te_X \cdot e_X = m_X^\circ \cdot e_X$ and let $X = (X, a)$ be a $(\mathbb{T}, \mathsf{V})$-category. Then the following assertions hold.*

(a) For all $\mathfrak{x} \in TX$ and $\varphi \in \mathsf{V}^{X^{\mathrm{op}}}$, $[\![a^{\mathrm{op}}, \hom_\xi]\!](Ty_0(\mathfrak{x}), \varphi) \ge \varphi(\mathfrak{x})$.

(b) Let $\mathfrak{x} \in TX$ such that $Ta \cdot e_{TX}(\mathfrak{x}, \mathfrak{x}) \ge k$. Then $[\![a^{\mathrm{op}}, \hom_\xi]\!](Ty_0(\mathfrak{x}), \varphi) \le \varphi(\mathfrak{x})$.

Proof. Let $\mathfrak{x} \in TX$ and $\varphi \in \mathsf{V}^{X^{\mathrm{op}}}$. As above, we obtain

$$\begin{aligned}
&[\![a^{\mathrm{op}}, \hom_\xi]\!](T_{\mathrm{o}}y(\mathfrak{x}), \varphi)\\
&= \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in T^2X, \mathfrak{W} \in T(TX \times \mathsf{V}^{X^{\mathrm{op}}})\\
&\qquad (T_{\mathrm{o}}\pi_1(\mathfrak{W}) = \mathfrak{Y} \,\&\, T_{\mathrm{o}}\pi_2(\mathfrak{W}) = T_{\mathrm{o}}y(\mathfrak{x})) \;\Rightarrow\; a^{\mathrm{op}}(\mathfrak{Y}, \mathfrak{x}) \otimes v \leq \hom(\xi \cdot T_{\mathrm{o}}\,\mathrm{ev}(\mathfrak{W}), \varphi(\mathfrak{y}))\}\\
&= \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX, \mathfrak{Y} \in T^2Y \,.\, Ta(\mathfrak{Y}, \mathfrak{x}) \otimes a^{\mathrm{op}}(\mathfrak{Y}, \mathfrak{y}) \otimes v \leq \varphi(\mathfrak{y})\}\\
&= \bigvee\{v \in \mathsf{V} \mid \forall \mathfrak{y} \in TX \;\; a^{\mathrm{op}} \cdot Ta^\circ(\mathfrak{x}, \mathfrak{y}) \otimes v \leq \varphi(\mathfrak{y})\}.
\end{aligned}$$

Furthermore, we have

$$a^{\mathrm{op}} \cdot Ta^\circ = e_{TX}^\circ \cdot Tm_X^\circ \cdot TTa^\circ \cdot Ta^\circ = e_{TX}^\circ \cdot Ta^\circ \leq m_X \cdot Ta^\circ.$$

Hence $\varphi(\mathfrak{x}) \leq [\![a^{\mathrm{op}}, \hom_\xi]\!](T_{\mathrm{o}}y_0(\mathfrak{x}), \varphi)$ and, if $k \leq Ta \cdot e_{TX}(\mathfrak{x}, \mathfrak{x}) = a^{\mathrm{op}} \cdot Ta^\circ(\mathfrak{x}, \mathfrak{x})$, we also have $[\![a^{\mathrm{op}}, \hom_\xi]\!](T_{\mathrm{o}}y_0(\mathfrak{x}), \varphi) \leq \varphi(\mathfrak{x})$. □

Example. Unlike y, the functor y_0 does not need to be full and faithful. In fact, consider $X = \mathbb{N}$ as a $(\mathbb{U}, 2)$-category, i.e. a topological space, equipped with the discrete topology $a = e_{\mathbb{N}}^\circ$. Then $\mathbb{N}^{\mathrm{op}}$ is the discrete space $\mathbb{N}^{\mathrm{op}} = (U\mathbb{N}, e_{U\mathbb{N}}^\circ)$. Let $\mathfrak{x}$ be a free ultrafilter on $\mathbb{N}$. Then, for each $\mathfrak{y} \in U\mathbb{N}$, $a^{\mathrm{op}} \cdot Ua^\circ(\mathfrak{x}, \mathfrak{y}) = e_{\mathbb{N}}^\circ \cdot Ue_{\mathbb{N}}(\mathfrak{x}, \mathfrak{y}) = \text{false}$ and therefore $Uy_0(\mathfrak{x}) \to \varphi$ for each $\varphi \in 2^{\mathbb{N}^{\mathrm{op}}}$. On the other hand, for $\varphi = a(-, x)$ (x any element of $\mathbb{N}$) we have $\varphi(\mathfrak{x}) = \text{false}$. In particular we see that $y_0 : \mathbb{N} \to 2^{\mathbb{N}^{\mathrm{op}}}$ is not full and faithful.

5 Examples

5.1 Ordered sets. Recall that $2\text{-}\mathsf{Cat} = \mathsf{Ord}$. Given an ordered set $X = (X, \leq)$, by Theorem 1.5 we have that a bimodule $\phi : 1 \,{-\!\!\circ\!\!\rightarrow}\, X$ is an order-preserving map $\phi : X \to 2$, while a bimodule $\psi : X \,{-\!\!\circ\!\!\rightarrow}\, 1$ is an order-preserving map $X^{\mathrm{op}} \to 2$. We can identify φ with the upclosed set $A = \varphi^{-1}(\text{true})$ and ψ with the downclosed set $B = \psi^{-1}(\text{true})$. Under this identification, $\varphi \dashv \psi$ translates to

$$A \cap B \neq \varnothing \qquad\qquad \text{and} \qquad\qquad \forall x \in A \;\; \forall y \in B \;\; y \leq x.$$

Then any $z \in A \cap B$ is simultaneously a smallest element of A and a largest element of B, therefore z represents $\varphi \dashv \psi$. Hence, by Proposition 1.6, each ordered set is Lawvere-complete. Note that the proof of Proposition 1.6 makes use of the Axiom of Choice. In fact, as pointed out in [6], here choice is essential.

Theorem. *In ZF, the following assertions are equivalent.*

(i) *Each ordered set is Lawvere-complete.*

(ii) *The Axiom of Choice.*

Proof. To see (ii)⇒(i), let $f : X \to Y$ be a surjective map. We equip Y with the discrete order Δ_Y and X with the kernel relation of f; then we have not only $f_* \dashv f^*$ but also $f^* \dashv f_*$. Hence there exists some $g : Y \to X$ which represents $f^* \dashv f_*$, and such g necessarily satisfies $f \cdot g = 1_Y$. □

5.2 Metric spaces. For $\mathsf{V} = \mathsf{P}_+$ we have $\mathsf{P}_+\text{-}\mathsf{Cat} \cong \mathsf{Met}$. The metric $d = \hom$ in P_+ is truncated minus, i.e. d(x,y)=0 if $x \geq y$ and $d(x,y) = y - x$ otherwise. Let $X = (X,d)$ be a metric space. A pair of adjoint bimodules $\varphi \dashv \psi$ corresponds to a pair of non-expansive maps $\varphi : X \to \mathsf{P}_+$ and $\psi : X^{\mathrm{op}} \to \mathsf{P}_+$ which satisfy

$$\inf_{x\in X} \varphi(x) + \psi(x) = 0 \qquad\qquad \text{and} \qquad\qquad \forall x, y \in X \;\; \psi(y) + \varphi(x) \geq d(y,x).$$

As observed in [28], pairs of adjoint bimodules on X correspond exactly to equivalence classes of Cauchy sequences. To see this, recall first that a sequence $s = (x_n)_{n\in\mathbb{N}}$ is called *Cauchy* if

$$\inf_{k\in\mathbb{N}} \sup_{n,n'\geq k} d(x_n, x_{n'}) = 0.$$

Given a Cauchy sequence $s = (x_n)_{n\in\mathbb{N}}$, we have

$$\inf_{m\in\mathbb{N}} \sup_{n\geq m} d(x_n, x) = \sup_{m\in\mathbb{N}} \inf_{n\geq m} d(x_n, x)$$

as well as

$$\inf_{m\in\mathbb{N}} \sup_{n\geq m} d(x, x_n) = \sup_{m\in\mathbb{N}} \inf_{n\geq m} d(x, x_n),$$

and s gives rise to non-expansive maps

$$\begin{aligned} \varphi_s : X &\to \mathsf{P}_+ \\ x &\mapsto \sup_{m\in\mathbb{N}} \inf_{n\geq m} d(x_n, x) \end{aligned} \qquad\qquad \text{and} \qquad\qquad \begin{aligned} \psi_s : X^{\mathrm{op}} &\to \mathsf{P}_+. \\ x &\mapsto \sup_{m\in\mathbb{N}} \inf_{n\geq m} d(x, x_n) \end{aligned}$$

One sees easily that $\varphi_s \dashv \psi_s$; moreover, two equivalent Cauchy sequences induce the same maps.

On the other hand, given an adjunction $\varphi \dashv \psi$, we may define $s = (x_n)_{n\in\mathbb{N}}$ such that $\varphi(x_n) + \psi(x_n) \leq \frac{1}{n}$, hence $d(x_n, x_m) \leq \frac{1}{n} + \frac{1}{m}$, and therefore s is a Cauchy sequence. Any two such sequences are equivalent. Furthermore, $\varphi \leq \varphi_s$ as well as $\psi \leq \psi_s$, therefore, since $\varphi \dashv \psi$ and $\varphi_s \dashv \psi_s$, we have even equality. Starting with a Cauchy sequence $s = (x_n)_{n\in\mathbb{N}}$, then for any sequence $t = (y_n)_{n\in\mathbb{N}}$ chosen for $\varphi \dashv \psi$ as above we have

$$\inf_{m\in\mathbb{N}} \inf_{k\in\mathbb{N}} \sup_{n\geq k} d(x_n, y_m) = 0 \qquad\qquad \text{and} \qquad\qquad \inf_{m\in\mathbb{N}} \inf_{k\in\mathbb{N}} \sup_{n\geq k} d(y_m, x_n) = 0,$$

hence s and t are equivalent. Finally, $s = (x_n)_{n\in\mathbb{N}}$ converges to x (i.e. s is equivalent to $(x)_{n\in\mathbb{N}}$) if and only if $\varphi_s \dashv \psi_s$ is represented by x.

The same argument also applies to the case $\mathsf{V} = \mathsf{P}_{\max}$: pairs of adjoint bimodules $\varphi \dashv \psi : 1 \to X$ with X an ultrametric space correspond precisely to Cauchy sequences in X, and convergence to representability.

Remark. A notion of non-symmetric Cauchy-sequence was introduced and studied in [4].

5.3 Topological spaces. We consider now $\mathbb{T} = \mathbb{U} = (U, e, m)$ the *ultrafilter monad* and $\mathsf{V} = 2$. As already stated, Proposition 3.1 describes our extension U in terms of $U_{\mathrm{o}} : \mathsf{Set} \to \mathsf{Set}$ (for a direct calculation of U, see [9, Example 6.4]). Then $(\mathbb{U}, 2)\text{-}\mathsf{Cat} = \mathsf{Top}$, as it was shown by Barr [1]. It is easily seen that the space $(2, \hom_\xi)$ is the Sierpinski space. By Theorem 3.3, a bimodule $\varphi : U1 \relbar\!\!\!\circ\!\!\!\rightarrow X$ from the one-element space 1 to a topological space X is essentially a continuous map $\varphi : X \to 2$ from X into the Sierpinski space 2, hence we can identify it with a closed subset $A \subseteq X$. A bimodule $\psi : UX \relbar\joinrel\mapstochar\joinrel\rightarrow 1$ is basically a map $\psi : UX \to 2$ such that $\mathcal{A} = \psi^{-1}(\text{true})$ is closed in $|X|$ as well as in X^{op}. The topology on $|X|$ is given by the Zariski closure, that is, $\mathfrak{x} \in UX$ is in the closure of $\mathcal{M} \subseteq UX$ if $\bigcap \mathcal{M} \subseteq \mathfrak{x}$. To understand the structure of X^{op}, observe first that the 2-category structure, i.e. the order on $M^\circ X$ is given by

$$\begin{aligned}
\mathfrak{x} \leq \mathfrak{y} &\iff \exists \mathfrak{X} \in U^2X \; m_X(\mathfrak{X}) = \mathfrak{x} \text{ and } \mathfrak{X} \to \mathfrak{y} \\
&\iff \forall A \in \mathfrak{x}, B \in \mathfrak{y} \;\; \exists \mathfrak{a} \in UA, y \in B \; \mathfrak{a} \to y \\
&\iff \forall A \in \mathfrak{x}, B \in \mathfrak{y} \;\; \overline{A} \cap B \neq \varnothing.
\end{aligned}$$

Denoting the filter base $\{\overline{A} \mid A \in \mathfrak{x}\}$ by $\bar{\mathfrak{x}}$, we have

$$\mathfrak{x} \leq \mathfrak{y} \iff \bar{\mathfrak{x}} \subseteq \mathfrak{y}.$$

Hence we can identify $\hat{X}$ with the set

$$\{\mathcal{A} \subseteq UX \,|\, \mathcal{A} \text{ is Zariski-closed and downclosed}\}.$$

By definition, the topology on $\hat{X}$ is the compact-open topology. Using the identification above, the sets

$$\{\mathcal{A} \in \hat{X} \,|\, \mathcal{A} \cap \mathcal{B} = \emptyset\},$$

with $\mathcal{B} \subseteq UX$ Zariski-closed, form a basis for the topology on $\hat{X}$. The Yoneda map $y_X : X \to \hat{X}$ sends $x \in X$ to $\{\mathfrak{x} \in UX \,|\, \mathfrak{x} \to x\}$. Hence bimodules $\psi : UX \relbar\joinrel\mapstochar\joinrel\rightarrow 1$ can be identified with subsets $\mathcal{A} \subseteq UX$ which are Zariski closed and down-closed for the order described above. Now $\varphi \dashv \psi$ translates as

$$\exists \mathfrak{x}_0 \in UX \;\; \mathfrak{x}_0 \in \mathcal{A} \;\&\; A \in \mathfrak{x}_0 \qquad\qquad \text{and} \qquad\qquad \forall \mathfrak{x} \in \mathcal{A}, x \in A \;\; \mathfrak{x} \to x.$$

Clearly, each $\mathfrak{x} \in \mathcal{A}$ converges to all points of A. On the other hand, for any $\mathfrak{x} \in UX$ with this property we have $\mathfrak{x} \leq \mathfrak{x}_0$ and therefore $\mathfrak{x} \in \mathcal{A}$. We conclude that

$$\mathcal{A} = \{\mathfrak{x} \in UX \mid \forall x \in A \;\; \mathfrak{x} \to x\}.$$

A closed subset $A \subseteq X$ admits an ultrafilter $\mathfrak{x}_0 \in UA$ which converges to all $x \in A$ if and only if $\{V \subseteq X \mid V \text{ open}, V \cap A \neq \varnothing\}$ is a filter base for x_0. In the language of closed sets this is expressed by saying that A is not the union of two proper closed subsets, i.e. A is *irreducible.* Finally, ψ (and hence φ) is representable if and only if $\mathfrak{x}_0$ can be chosen principal, that is, if and only if there exists some point $x_0 \in A$ which converges to all $x \in A$. In conclusion, we have

Theorem. *The following assertions are equivalent for a topological space X.*

(i) *X is Lawvere-complete.*

(ii) *Each irreducible closed subset $A \subseteq X$ is of the form $A = \overline{\{x\}}$ for some $x \in A$, i.e. X is weakly sober.*

5.4 Approach spaces. Recall that $\mathsf{App} = (\mathbb{U}, \mathsf{P}_{\!+})\text{-}\mathsf{Cat}$ is the category of approach spaces and non-expansive maps. We fix an approach space $X = (X, a)$. As above, a bimodule $\varphi : U1 \,{-\!\!\circ\!\!\rightarrow}\, X$ is a non-expansive map $\varphi : X \to \mathsf{P}_{\!+}$, by Theorem 3.3. There is a bijective correspondence between maps $\varphi : X \to \mathsf{P}_{\!+}$ and families $(A_v)_{v \in \mathsf{P}_{\!+}}$ of subsets $A_v \subseteq X$ satisfying

$$A_v = \bigcap_{u>v} A_u, \tag{5}$$

where $\varphi \mapsto (\varphi^{-1}([0, v]))_{v \in \mathsf{P}_{\!+}}$ and a family $(A_v)_{v \in \mathsf{P}_{\!+}}$ defines the map $x \mapsto \inf\{v \in \mathsf{P}_{\!+} \mid x \in A_v\}$. Under this bijection, non-expansive maps correspond precisely to those families $(A_v)_{v \in \mathsf{P}_{\!+}}$ which satisfy in addition

$$\forall u, v \in \mathsf{P}_{\!+}\ (A_u)^{(v)} \subseteq A_{u+v}), \tag{6}$$

where $A^{(v)} = \{x \in X \,|\, d(A, x) \le v\}$ and $d(A, x) = \inf\{a(\mathfrak{x}, x) \mid \mathfrak{x} \in UA\}$, for $v \in V$, $A \subseteq X$.

We may think of the family $A = (A_v)_{v \in \mathsf{P}_{\!+}}$ satisfying (5) as a *variable set*[1]; we call A *closed* if it satisfies (6). Now it is not difficult to see that a right adjoint $\psi : X \,{-\!\!\circ\!\!\rightarrow}\, 1$ to $\varphi : 1 \,{-\!\!\circ\!\!\rightarrow}\, X$ is determined by the variable set $\mathcal{A} = (\mathcal{A}_v)_{v \in \mathsf{P}_{\!+}}$ given by

$$\mathcal{A}_v = \{\mathfrak{x} \in UX \mid \forall u \in \mathsf{P}_{\!+}\ \ \forall x \in A_u\ \ a(\mathfrak{x}, x) \le u + v\},$$

for each $v \in \mathsf{P}_{\!+}$. Furthermore, given $\varphi : 1 \,{-\!\!\circ\!\!\rightarrow}\, X$, the variable set $\mathcal{A}$ defined as above corresponds to a right adjoint of φ if and only if

$$\forall u \in \mathsf{P}_{\!+}\ (u > 0 \Rightarrow UA_u \cap \mathcal{A}_u \neq \varnothing). \tag{7}$$

In analogy to the situation in Top, we call a variable set A *irreducible* if it satisfies (7). Finally, we remark that the bimodule $\varphi : 1 \,{-\!\!\circ\!\!\rightarrow}\, X$ is represented by $x \in X$ precisely if the corresponding variable set A is of the form

$$A_v = \{y \in X \mid d(x, y) \le v\},$$

for each $v \in \mathsf{P}_{\!+}$. Naturally, we say that such a variable set is *representable* (by x).

Theorem. *The following assertions are equivalent for an approach space X.*

(i) *X is Lawvere-complete.*

(ii) *Each irreducible closed variable set A is representable.*

We point out that this setting satisfies the conditions of Theorem 3.4, therefore it assures that $\mathsf{P}_{\!+}$ is Lawvere-complete.

The structure on X^{op} is based on the $\mathsf{P}_{\!+}$-category $M^{\circ}(X)$, which is the metric space (UX, d), with

$$d(\mathfrak{x}, \mathfrak{y}) = \inf\{v \in [0, \infty] \,|\, \{A^{(v)} \,|\, A \in \mathfrak{x}\} \subseteq \mathfrak{y}\}.$$

Remark. The notion of approach frame and its connection with approach spaces was recently studied by Christophe Van Olmen in his PhD thesis [33]. In particular, the concept of *sober approach space* as a fixed point of the dual adjunction between App and the category AFrm of approach frames and homomorphisms was introduced. As confirmed by the author of [33], these are precisely the approach spaces where each irreducible closed variable set is uniquely representable.

[1] In fact, we may consider $A : \mathsf{P}_{\!+} \to \mathsf{Set}$ as a sheaf where, for each $u \in \mathsf{P}_{\!+}$, $\{v < u\}$ is a cover of u.

6 Lawvere-complete quasi-uniform spaces

6.1 Cauchy-complete quasi-uniform spaces. We recall that a *quasi-uniformity* U on a set X is a set of relations on X such that:

$$\forall u \in U \ \ \Delta \subseteq u;$$

$$\forall u \in U \ \ \exists v \in U \ \ v \cdot v \subseteq u.$$

The pair (X, U) is called a *quasi-uniform space*; it is a *uniform space* when, for all $u \in U$, $u^{-1} \in U$. Given quasi-uniform spaces (X, U) and (Y, V), a map $f : X \to Y$ is *uniformly continuous* if

$$\forall v \in V \ \ \exists u \in U \ \ \forall x, y \in X \ \ x\, u\, y \ \Rightarrow \ f(x)\, v\, f(y).$$

Definition. Let (X, U) be a quasi-uniform space.

1. A pair $(\mathfrak{f}, \mathfrak{g})$ is a *filter* on (X, U) if $\mathfrak{f}$ and $\mathfrak{g}$ are filters on X such that

$$\forall F \in \mathfrak{f} \ \ \forall G \in \mathfrak{g} \ \ F \cap G \neq \varnothing.$$

2. A filter $(\mathfrak{f}, \mathfrak{g})$ on (X, U) is a *Cauchy filter* if

$$\forall u \in U \ \ \exists F \in \mathfrak{f} \ \ \exists G \in \mathfrak{g} \ \ F \times G \subseteq X_u := \{(x, x') \mid x\, u\, x'\}.$$

3. A filter $(\mathfrak{f}, \mathfrak{g})$ on (X, U) *converges to* $x_0 \in X$ if

$$\forall u \in U \ \ \exists F \in \mathfrak{f} \ \ \exists G \in \mathfrak{g} \ \ F \times G \subseteq X_{-ux_0} \times X_{x_0u-},$$

where $X_{-ux_0} := \{x \in X \mid x\, u\, x_0\}$ and $X_{x_0u-} := \{x \in X \mid x_0\, u\, x\}$.

Lemma. *Given a quasi-uniformity U in X and $x_0 \in X$, the neighbourhood filter of x_0*

$$(\{X_{-ux_0} \mid u \in U\}, \{X_{x_0u-} \mid u \in U\})$$

is a minimal Cauchy filter on (X, U).

Proposition. *For a quasi-uniform space (X, U), the following conditions are equivalent.*

(i) *Every Cauchy filter converges.*

(ii) *Every minimal Cauchy filter is the neighbourhood filter of a point x_0.*

A quasi-uniform space is said to be *Cauchy-complete* if it satisfies any of the equivalent conditions of the Proposition.

For further information see [19] and [20].

6.2 Quasi-uniform spaces as lax algebras. In order to describe quasi-uniform spaces as lax algebras, we turn back to the setting described in [9] and substitute the bicategory $\mathsf{V}\text{-}\mathsf{Rel}$ of 2.1 by the bicategory Y having sets as objects and (possibly improper) filters on $\mathsf{Rel}(X, Y)$ as morphisms, where Rel is the bicategory of relations. The composition of two filters $R : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and $S : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ is the filter obtained by pointwise composition of relations $R \cdot S = \{s \cdot r \,|\, s \in S$ and $r \in R\}$, while $R \leq R'$ whenever $R' \subseteq R$ (as sets).

We define a *lax algebra* now exactly like a V-category: it is a Y-morphism $A : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ such that

$$1_X \leq A \quad \text{and} \quad A \cdot A \leq A,$$

or, equivalently,

$$\forall x \in X \;\; \forall a \in A \;\; x\,a\,x \quad \text{and} \quad \forall a \in A \; \exists a' \in A \;\; a' \cdot a' \leq a.$$

A *lax morphism* $f : (X, A) \to (Y, B)$ between lax algebras is a map $f : X \to Y$ such that $f \cdot A \leq B \cdot f$, i.e.

$$\forall b \in B \;\; \exists a \in A \;\; f \cdot a \leq b \cdot f.$$

It was shown in [9, Theorem 3.6] that this category of lax algebras and lax morphisms is equivalent to the category of quasi-uniform spaces and uniformly continuous maps.

6.3 Adjoint pairs of bimodules in quasi-uniform spaces. A *bimodule* $\Psi : (X, A) \relbar\joinrel\circ\joinrel\rightarrow (Y, B)$ between lax algebras is a Y-morphism $\Psi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ such that $\Psi \cdot A \leq \Psi$ and $B \cdot \Psi \leq \Psi$. As in the context of V-categories, A and B act as identities for the composition with bimodules, so that a pair of bimodules $(\Phi : (Y, B) \relbar\joinrel\circ\joinrel\rightarrow (X, A), (\Psi : (X, A) \relbar\joinrel\circ\joinrel\rightarrow (Y, B))$ is an *adjoint pair*, with $\Phi \dashv \Psi$, if $B \leq \Psi \cdot \Phi$ and $\Phi \cdot \Psi \leq A$. As before, every lax morphism $f : (X, A) \to (Y, B)$ defines a pair of adjoint bimodules $(f_* = B \cdot f : (X, A) \relbar\joinrel\circ\joinrel\rightarrow (Y, B), f^* = f^\circ \cdot A : (Y, B) \relbar\joinrel\circ\joinrel\rightarrow (X, A))$. It is easy to check that Proposition 2.7 is still valid in this context.

Proposition. *For a lax algebra (X, A), the following conditions are equivalent.*

(i) *Each pair of adjoint bimodules $(\Phi : (Y, B) \relbar\joinrel\circ\joinrel\rightarrow (X, A)) \dashv (\Psi : (X, A) \relbar\joinrel\circ\joinrel\rightarrow (Y, B))$ is induced by a lax morphism $(Y, B) \to (X, A)$.*

(ii) *Each pair of adjoint bimodules $(\Phi : 1 \relbar\joinrel\circ\joinrel\rightarrow (X, A)) \dashv (\Psi : (X, A) \relbar\joinrel\circ\joinrel\rightarrow 1)$ is induced by a lax morphism $1 \to (X, A)$ (or simply a map).*

Theorem. *For Y-morphisms $\Phi : 1 \relbar\joinrel\mapstochar\joinrel\rightarrow X$ and $\Psi : X \relbar\joinrel\mapstochar\joinrel\rightarrow 1$, the following conditions are equivalent.*

(i) $\Phi \dashv \Psi$.

(ii) $(\{X_{-\psi\star} \,|\, \psi \in \Psi\}, \{X_{\star\varphi-} \,|\, \varphi \in \Phi\})$ *is a minimal Cauchy filter on (X, A).*

Proof. The conditions $1 \leq \Psi \cdot \Phi$ and $\Phi \cdot \Psi \leq A$ read as

$$\forall \psi \in \Psi \;\; \exists \varphi \in \Phi \;\; X_{\star\varphi-} \cap X_{-\psi\star} \neq \varnothing,$$

$$\forall a \in A \;\; \exists \varphi \in \Phi \;\; \exists \psi \in \Psi \;\; X_{-\psi\star} \times X_{\star\varphi-} \subseteq X_a,$$

where the former condition means that $(\{X_{-\psi\star} \,|\, \psi \in \Psi\}, \{X_{\star\varphi-} \,|\, \varphi \in \Phi\})$ is a filter, while the latter one means that it is Cauchy.

(i) $\Rightarrow$ (ii): It remains to be shown that this Cauchy filter is minimal. Let $(\mathfrak{f}, \mathfrak{g})$ be a filter contained in it. If $\mathfrak{f} \subsetneq \{X_{-\psi\star} \,|\, \psi \in \Psi\}$, i.e. if there exists $\psi \in \Psi$ such that $X_{-\psi\star} \notin \mathfrak{f}$, then there exist $a \in A$ and $\psi' \in \Psi$ with $\psi' \cdot a = \psi$, because ψ is a bimodule, hence a and ψ' are such that $\bigcup\limits_{x' \in X_{-\psi'\star}} X_{-ax'} \notin \mathfrak{f}$. Therefore

$$\forall F \in \mathfrak{f} \;\; \exists x \in F \;\; \forall x' \in X_{-\psi'\star} \;\; (x, x') \notin X_a.$$

Moreover, since

$$\forall G \in \mathfrak{g} \;\; G \in \{X_{\star\varphi-} \,|\, \varphi \in \Phi\} \;\Rightarrow\; \forall G \in \mathfrak{g} \;\; \exists y \in X_{-\psi'\star} \cap G,$$

we obtain

$$\forall F \in \mathfrak{f} \;\; \forall G \in \mathfrak{g} \;\; \exists x \in F \;\; \exists y \in G \;\; (x, y) \notin X_a,$$

that is $(\mathfrak{f}, \mathfrak{g})$ is not a Cauchy filter.

(ii) $\Rightarrow$ (i): Let $\Phi : 1 \relbar\!\!\!\circ\!\!\!\rightarrow (X, A)$ and $\Psi : (X, A) \relbar\!\!\!\circ\!\!\!\rightarrow 1$ be a pair of bimodules and consider $(\{X_{-\psi\star} \,|\, \psi \in \Psi\}, \{X_{\star\varphi-} \,|\, \varphi \in \Phi\})$. We concluded already that the adjunction conditions are equivalent to this pair being a Cauchy filter. But we did not show yet that Φ and Ψ are bimodules. For any $a \in A$,

$$(\{\bigcup_{x \in X_{-\psi\star}} X_{-ax} \,|\, \psi \in \Psi,\, a \in A\},\; \{\bigcup_{y \in X_{\star\varphi-}} X_{ya-} \,|\, \varphi \in \Phi, a \in A\})$$

is a Cauchy filter contained in the former one, as we show next. First,

$$\bigcup_{x \in X_{-\psi\star}} X_{-ax} \;\cap\; \bigcup_{y \in X_{\star a-}} X_{ya-} \;\supseteq\; X_{-\psi\star} \cap X_{\star\varphi-} \neq \varnothing.$$

To prove the other condition, let $a \in A$, and consider $b \in A$ such that $b \cdot b \cdot b \leq a$. There exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that $X_{-\psi\star} \times X_{\star\varphi-} \subseteq X_b$, and this implies that

$$\bigcup_{x \in X_{-\psi\star}} X_{-bx} \times \bigcup_{y \in X_{\star\varphi-}} X_{yb-} \subseteq X_a,$$

since

$$x' \in \bigcup_{x \in X_{-\psi\star}} X_{-bx} \;\Rightarrow\; \exists x \in X_{-\psi\star} \;\; (x', x) \in X_b,$$

$$y' \in \bigcup_{y \in X_{\star\varphi-}} X_{yb-} \;\Rightarrow\; \exists y \in X_{\star\varphi-} \;\; (y, y') \in X_b;$$

hence, since also $(x, y) \in X_b$, we conclude that $(x', y') \in X_a$ as claimed. □

6.4 Lawvere-complete=Cauchy-complete. It is now straightforward to prove that the two notions of completeness coincide.

Theorem. *For a quasi-uniform space (X, A) the following conditions are equivalent.*

(i) *(X, A) is a Lawvere-complete lax algebra.*

(ii) (X, A) *is a Cauchy-complete quasi-uniform space.*

Proof. (i) $\Rightarrow$ (ii): Each minimal Cauchy filter on (X, A) defines an adjoint pair of bimodules $(\Phi : 1 \multimapdotbothvert (X, A)) \dashv (\Psi : (X, A) \multimapdotbothvert 1)$, which, by (i), is induced by a map $f : 1 \to X$, $\star \mapsto x_0$. Hence $\Phi = \{\varphi_b = b \cdot f \,|\, b \in B\}$ and $\Psi = \{\psi_b = f^\circ \cdot b \,|\, b \in B\}$. Moreover, $x \in X_{\star\varphi_b -}$ exactly when $b(x_0, x) = \top$, that is $X_{\star\varphi_b -} = X_{x_0 b -}$, and $x \in X_{-\psi_b \star}$ exactly when $b(x, x_0) = \top$, which means $X_{-\psi_b\star} = X_{-bx_0}$.

(ii) $\Rightarrow$ (i): Given an adjoint pair of bimodules $(\Phi : 1 \multimapdotbothvert (X, A)) \dashv (\Psi : (X, A) \multimapdotbothvert 1)$, by (ii) the minimal Cauchy filter it induces is the neighbourhood filter of a point x_0. It is straightforward to check that $\Phi = A \cdot f$ and $\Psi = f^\circ \cdot A$ for $f : 1 \to X$, $\star \mapsto x_0$. $\square$

Final remark. The results of this section can be investigated in the more general setting introduced in [16], i.e., in proalgebras; here, for simplicity, we decided to state them only at the level of quasi-uniform structures, which are proalgebras for the identity monad.

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