Duality-TV

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Buon compleano e tanti auguri!

Disclaimer

The following program contains ultrafilters, modules, presheafs and incorrect english. Viewer discretion is advised.

The motivating example (Rosebrugh and Wood, 1994; Raney, 1952):

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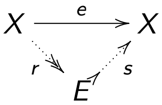

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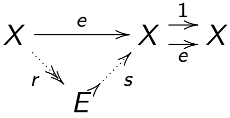

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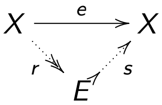

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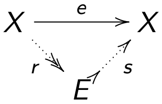

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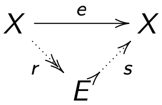

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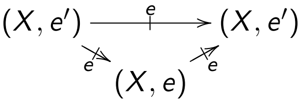

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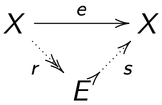

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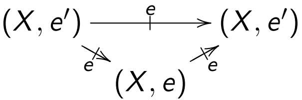

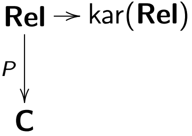

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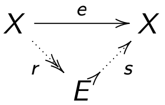

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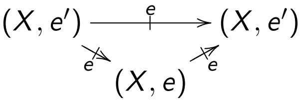

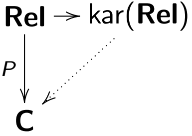

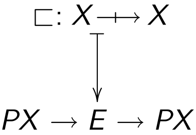

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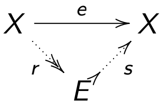

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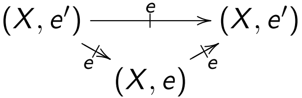

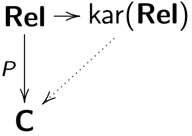

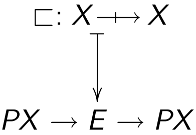

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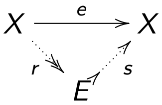

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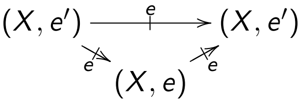

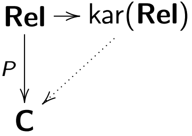

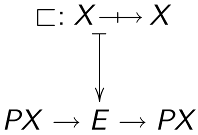

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For X (cd), $t_X : X \to \mathbf{2}^{X^{\mathrm{op}}}$ gives $\ll : X ⇸ X$ idempotent, hence $(X, \ll)$ is in $\mathsf{kar}(\mathbf{Rel})$.

Hence one has:

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\uparrow & & \uparrow \\
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Note: $\varphi : X \multimap\!\!\!\!\circ\, Y$ corresponds to $\ulcorner \varphi \urcorner : Y \to \mathbf{2}^{X^{\mathrm{op}}}$.

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Note: $\varphi : X \circ\!\!\rightarrow Y$ corresponds to $\ulcorner \varphi \urcorner : Y \to \mathbf{2}^{X^{\mathrm{op}}}$.

$\mathbf{Rel} \hookrightarrow \mathbf{Mod} \hookrightarrow \mathsf{kar}(\mathbf{Rel})$, hence $\mathsf{kar}(\mathbf{Mod}) \cong \mathsf{kar}(\mathbf{Rel})$.

One also has:

$$\mathsf{kar}(\mathbf{Mod})^{\mathrm{op}} \cong \mathbf{CCD}_{\mathrm{sup}}$$

For X, Y in **Ord**, a *module* $\varphi : X \multimap Y$ is a relation $\varphi : X \relbar\!\!\!\mapsto Y$ where

- $\varphi : X^{\mathrm{op}} \times Y \to \mathbf{2}$ is monotone, or
- $(x \leq x' \,\&\, x' \,\varphi\, y' \,\&\, y' \leq y) \;\Rightarrow\; x \,\varphi\, y$, or
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Rel $\hookrightarrow$ **Mod** $\hookrightarrow$ kar(**Rel**), hence kar(**Mod**) $\cong$ kar(**Rel**).

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One also has:

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One also has:

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For X, Y in **Ord**, a *module* $\varphi : X \circ\!\!\!\longrightarrow Y$ is a relation $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ where

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\uparrow & & \uparrow \\
\mathbf{Ord} \cong \mathsf{map}(\mathbf{Mod}) & \longrightarrow & \mathbf{Tal}^{\mathrm{op}}
\end{array}$$

- For a monotone map $f : X \to Y$:

$$f_* \dashv f^* \text{ where } \begin{cases} f_* : X \multimap Y, \; x \, f_* \, y \text{ if } f(x) \leq y, \\ f^* : Y \multimap X, \; y \, f^* \, x \text{ if } y \leq f(x). \end{cases}$$

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- $\mathbb{P} = (P, y, m)$ is the down-set monad on **Ord** where $y_X(x) = x^*$ and $m_X(\Psi) = \bigcup \Psi$.

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From **Ord** to **Met**: substitute **2** by $[0, \infty]$, & by $+$, true by 0, $x \Rightarrow y$ sometimes by $x \geqslant y$ and sometimes by $y - x$ (truncated minus), $\exists$ by inf, $\forall$ by sup, and so on.

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- Metric space: set X with $a : X \times X \to [\infty, 0]$ s.t.

$$0 \geqslant a(x, x), \qquad\qquad a(x, y) + a(y, z) \geqslant a(x, z).$$

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- As before:
  - each contraction map $f : X \to Y$ induces modules $f_* \dashv f^*$ where $f_*(x, y) = b(f(x), y)$ and $f^*(y, x) = b(y, f(x))$,

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  - a metric space is cocomplete if $\mathrm{Sup}_X \dashv \mathscr{y}_X$,
  - a metric space is (cd) if $t_X \dashv \mathrm{Sup}_X \dashv \mathscr{y}_X$.

Again, one has:

$$
\begin{array}{ccc}
\mathsf{kar}(\mathbf{NRel}) & \overset{\cong}{\rightleftarrows} & \mathbf{CDMet}_{\mathrm{sup}}^{\mathrm{op}} \\
\uparrow & & \uparrow \\
\mathbf{NRel} & \longrightarrow & \mathbf{CAMet}_{\mathrm{sup}}^{\mathrm{op}} \\
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\uparrow & & \uparrow \\
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\uparrow & & \uparrow \\
\mathbf{Met}_{\mathrm{cc}} \cong \mathsf{map}(\mathbf{NMod}) & \longrightarrow & \mathbf{TAMet}^{\mathrm{op}}
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Lawvere, 1973: For a metric space X, TFAE:

1. (i) X is Cauchy complete.
2. (ii) Every left adjoint module $\varphi : Y \,{-\!\!\circ\!\!\rightarrow}\, X$ is of the form $\varphi = f_*$.

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(vi) $y_X : X \to \widetilde{X}$ is an equivalence.

Theorem (Rosebrugh and Wood, 2004)

Let $\mathbb{T}$ *be a monad on* **C** *Cauchy-complete. Then*

$$\mathsf{kar}(\mathbf{C}_{\mathbb{T}}) \cong \mathsf{Spl}(\mathbf{C}^{\mathbb{T}}).$$

R. Rosebrugh and R.J. Wood.
Split structures.
Theory Appl. Categ. **13** (2004), 172–183.

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For $\mathbb{P}$ the power-set monad on **Set** resp. the down-set monad on **Ord**

$$\mathbf{Rel}^{\mathrm{op}} \cong \mathbf{Set}_{\mathbb{P}}, \qquad \mathbf{Mod}^{\mathrm{op}} \cong \mathbf{Ord}_{\mathbb{P}}, \qquad \mathbf{Sup} \cong \mathbf{Set}^{\mathbb{P}} \cong \mathbf{Ord}^{\mathbb{P}}.$$

Topological and approach spaces are categories. . .

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- Topological space: set X with $\to : UX \times X \to \mathbf{2}$ s.t.

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- Approach space: set X with $\lambda : UX \times X \to [0, \infty]$ s.t.

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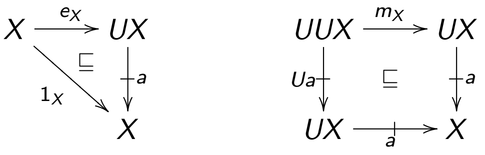
- Both axioms read as (where $a : UX \looparrowright X$)



$$1_X \sqsubseteq a \cdot e_X \qquad a \cdot Ua \sqsubseteq a \cdot m_X$$

where $\sqsubseteq$ stands either for $\subseteq$ or $\geqslant$.

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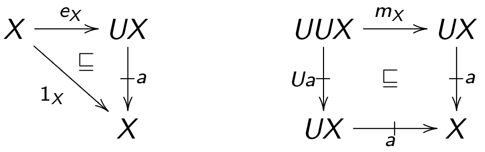

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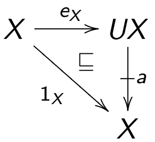

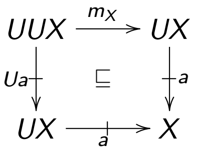

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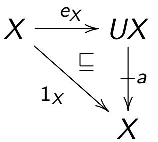

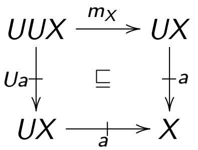

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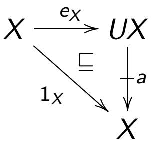

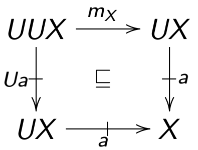

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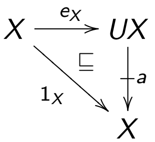

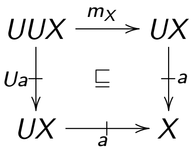

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- We define now $(-)^{\mathrm{op}} : \mathbf{Top} \to \mathbf{Top}$ and $(-)^{\mathrm{op}} : \mathbf{App} \to \mathbf{App}$ by

$$\begin{array}{ccc} \mathbf{Top} & \dashrightarrow & \mathbf{Top} \\ {\scriptstyle M}\downarrow & & \uparrow{\scriptstyle K} \\ \mathbf{OrdCH} & \xrightarrow[(-)^{\mathrm{op}}]{} & \mathbf{OrdCH} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbf{App} & \dashrightarrow & \mathbf{App} \\ {\scriptstyle M}\downarrow & & \uparrow{\scriptstyle K} \\ \mathbf{MetCH} & \xrightarrow[(-)^{\mathrm{op}}]{} & \mathbf{MetCH} \end{array}$$

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\uparrow & & \uparrow \\
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\mathrm{kar}(\mathbb{U}\text{-}\mathbf{Mod}) & \overset{\cong}{\rightleftarrows} & \mathbf{CDTop}^{\mathrm{op}}_{\mathrm{sup}}/\mathbf{CDApp}^{\mathrm{op}}_{\mathrm{sup}} \\
\uparrow & & \uparrow \\
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- objects: (X, θ) where $\theta \circ \theta = \theta$,
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What about **CDTop**? One has (recall $PX \cong F\mathcal{O}X$)

X is (cd) iff X is the filter space of some frame.

$$F : \mathbf{Frm} \to \mathbf{CDTop}, \qquad L \mapsto \{\text{filters on } L\},$$
$$\mathrm{Pt} : \mathbf{CDTop} \to \mathbf{Frm}, \qquad X \mapsto \{x \in X \mid \downarrow x \text{ is open}\},$$

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One has:

$$\mathbf{Inf} \to \mathrm{map}(\mathrm{kar}(\mathbb{U}\text{-}\mathbf{Rel}))$$
$$f \mapsto f_+$$

and $\mathrm{map}(\mathrm{kar}(\mathbb{U}\text{-}\mathbf{Rel}))(Y, X) \cong \mathbf{Inf}(Y, S(X, \theta))$.

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For $\mathbb{T} = \mathbb{U}$, $\varphi : \mathbf{V}^X \to \mathbf{V}$ preserving infima and (co)tensors:

$$\varphi \text{ preserves } \mathbb{U}\text{-suprema} \iff \varphi \text{ preserves finite suprema.}$$

Let Φ-**Mod** be a (non-full) subcategory of $\mathbb{U}$-**Mod** with

- $f^* \in \Phi$-**Mod**, for every f in **Top**/**App**,
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\uparrow & & \uparrow \\
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