

# Completely distributive spaces

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CT 2009

$$\text{Ord} \begin{array}{c} \xrightarrow{\text{hom}(-,2)} \\ \perp \\ \xleftarrow{\text{hom}(-,2)} \end{array} \text{CCD}^{\text{op}}$$

$$\text{Ord} \xrightarrow{(-)^{\text{op}}} \text{Ord} \begin{array}{c} \xrightarrow{\text{hom}(-,2)} \\ \perp \\ \xleftarrow{\text{hom}(-,2)} \end{array} \text{CCD}^{\text{op}}$$

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$$X \longmapsto 2^{X^{\text{op}}}$$

$$L \longmapsto [L, 2]^{\text{op}}$$



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ccd-lattice:

$$\begin{array}{ccc} & \lceil \ll \rceil & \\ & \curvearrowright & \\ L & \xleftrightarrow{\text{Sup}} & PL \\ & \curvearrowleft & \\ & \lfloor \gg \rfloor & \\ & y_L & \end{array}$$

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ccd-lattice:

$$A \xrightarrow{i} L \begin{array}{c} \begin{array}{c} \lceil \ll \rceil \\ \curvearrowright \\ \perp \\ \text{Sup} \\ \perp \\ \curvearrowleft \\ \lfloor \gg \rfloor \end{array} \\ \text{PL} \end{array}$$

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Distributor:

$$\varphi : X \multimap Y \Leftrightarrow \varphi : X^{\text{op}} \times Y \longrightarrow 2 \text{ monotone} \Leftrightarrow b \cdot \varphi = \varphi = \varphi \cdot a.$$

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### Distributor:

$\varphi : X \multimap Y \Leftrightarrow \varphi : X^{\text{op}} \times Y \longrightarrow 2$  monotone  $\Leftrightarrow b \cdot \varphi = \varphi = \varphi \cdot a$ .

For  $f : X \longrightarrow Y$ :  $(f_* : X \multimap Y) \dashv (f^* : Y \multimap X)$ .

$$\text{Ord} \xrightarrow{(-)^{\text{op}}} \text{Ord} \begin{array}{c} \xrightarrow{\text{hom}(-,2)} \\ \perp \\ \xleftarrow{\text{hom}(-,2)} \end{array} \text{CCD}^{\text{op}}$$

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About the units:

For  $PX := \text{Dist}(X, 1)$ :  $X \xrightarrow{y_X} PX \xrightarrow{y_{PX}} PPX$

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For  $PX := \text{Dist}(X, 1)$ :

$$X \xrightarrow{y_X} PX \begin{array}{c} \xleftarrow{(-) \cdot y_{X^*}} \\ \perp \\ \xrightarrow{y_{PX}} \end{array} PPX$$

$$\text{Ord} \xrightarrow{(-)^{\text{op}}} \text{Ord} \begin{array}{c} \xrightarrow{\text{hom}(-,2)} \\ \perp \\ \xleftarrow{\text{hom}(-,2)} \end{array} \text{CCD}^{\text{op}}$$

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For  $L$  in CCD:  $L \xrightarrow{\varepsilon_L} PA, x \mapsto \{z \in A \mid z \leq x\};$

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$$\text{Ord} \begin{array}{c} \xrightarrow{\cong} \\ \xleftarrow{\cong} \end{array} \text{Tal}^{\text{op}}$$

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$$\begin{array}{ccc}
 \text{Ord} & \xrightarrow{\cong} & \text{Tal}^{\text{op}} \\
 \downarrow & & \downarrow \\
 \text{Dist} & \xrightarrow{\cong} & \text{Tal}_{\text{sup}}^{\text{op}}
 \end{array}$$

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$  \begin{array}{ccc}  \vdash & \uparrow & \vdash \\  \swarrow & \vdash \cdot f_* \vdash & \searrow \\  & \uparrow & \\  & \vdash & \\  \swarrow & & \searrow  \end{array}  $	

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M. Barr (1970):

Topological space = set  $X$  with  $a : UX \times X \rightarrow 2$  such that

$$\dot{x} \rightarrow x, \quad (\mathfrak{X} \rightarrow \mathfrak{r} \ \& \ \mathfrak{r} \rightarrow x) \Rightarrow (m_X(\mathfrak{X}) \rightarrow x).$$

# Topological spaces

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From here we have two choices:

- $X \mapsto 2^X$  leads to  $\text{Top} \begin{array}{c} \xrightarrow{\text{hom}(-,2)} \\ \perp \\ \xleftarrow{\text{hom}(-,2)} \end{array} \text{Frm}^{\text{op}}$

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- $X \mapsto 2^{UX}$  leads to ...?...

# The dictionary

# The dictionary

relation $r : X \times Y \rightarrow 2$	
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# The dictionary

$$V = (V, \otimes, k)$$

V-relation  $r : X \times Y \longrightarrow V$

# The dictionary

$$\mathbb{V} = (\mathbb{V}, \otimes, k)$$

$$\mathcal{T} = (\mathbb{T}, \mathbb{V}, \xi : T\mathbb{V} \rightarrow \mathbb{V})$$

V-relation  $r : X \times Y \longrightarrow \mathbb{V}$

$\mathcal{T}$ -relation  $r : TX \times Y \longrightarrow \mathbb{V}$

# The dictionary

$$V = (V, \otimes, k)$$

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relational composition

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$V = (V, \otimes, k)$	$\mathcal{T} = (\mathbb{T}, V, \xi : TV \rightarrow V)$
$r : X \dashrightarrow Y$ relational composition	$r : X \dashrightarrow Y$ Kleisli composition

## Kleisli composition:

- $(TX \dashrightarrow Y, TY \dashrightarrow Z) \mapsto (TX \xrightarrow{m_X^\circ} TTX \dashrightarrow TY \dashrightarrow Z)$
- One has  $e_Y^\circ \circ r \geq r$  and  $r \circ e_X^\circ = r$ .

# The dictionary

$\mathcal{V} = (\mathbb{V}, \otimes, k)$	$\mathcal{T} = (\mathbb{T}, \mathbb{V}, \xi : TV \rightarrow \mathbb{V})$
$r : X \multimap Y$ relational composition V-category $X = (X, a)$	$r : X \multimap Y$ Kleisli composition

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- $(TX \xrightarrow{r} Y, TY \xrightarrow{s} Z) \mapsto (TX \xrightarrow{m_X^\circ} TTX \xrightarrow{Tr} TY \xrightarrow{s} Z)$
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V-category:  $a : X \multimap X$  with  $1_X \leq a$  and  $a \cdot a \leq a$ .

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$\mathbb{V} = (\mathbb{V}, \otimes, k)$	$\mathcal{T} = (\mathbb{T}, \mathbb{V}, \xi : T\mathbb{V} \rightarrow \mathbb{V})$
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**V-category:**  $a : X \dashrightarrow X$  with  $1_X \leq a$  and  $a \cdot a \leq a$ .

**T-category:**  $a : X \dashrightarrow X$  with  $e_X^\circ \leq a$  and  $a \circ a \leq a$

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$r : X \dashrightarrow Y$ relational composition V-category $X = (X, a)$ V-distributor $\varphi : X \dashv\vdash Y$	$r : X \dashrightarrow Y$ Kleisli composition $\mathcal{T}$ -category $X = (X, a)$

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## Recall:

- $\varphi : X \dashv\vdash Y \iff \varphi : X^{\text{op}} \otimes Y \longrightarrow V$  is a V-functor  
 $\iff \ulcorner \varphi \urcorner : Y \longrightarrow V^{X^{\text{op}}}$  is a V-functor.

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# The dictionary

$\mathbb{V} = (\mathbb{V}, \otimes, k)$	$\mathcal{T} = (\mathbb{T}, \mathbb{V}, \xi : T\mathbb{V} \rightarrow \mathbb{V})$
$r : X \dashrightarrow Y$ relational composition	$r : X \dashrightarrow Y$ Kleisli composition
V-category $X = (X, a)$	$\mathcal{T}$ -category $X = (X, a)$
V-distributor $\varphi : X \dashv\vdash Y$	$\mathcal{T}$ -distributor $\varphi : X \dashv\vdash Y$
$X^{\text{op}}$	$?? = (TX, ?)$ $\otimes$ -exp.
Yoneda	??

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$$M : \mathcal{T}\text{-Cat} \longrightarrow V\text{-Cat}^{\mathbb{T}}.$$

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$$\text{Here: } TX \xrightarrow{m_X \circ} TTX \xrightarrow{Ta} TX$$

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Hence we define:

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- $K : \mathbf{V}\text{-Cat}^{\mathbb{T}} \longrightarrow \mathcal{T}\text{-Cat}$  is strict monoidal.
- $\mathbf{V}\text{-Cat}^{\mathbb{T}} \cong \mathcal{T}\text{-Cat}^{\mathbb{T}}$ ; ( $\mathcal{T}\text{-Cat}^{\mathbb{T}}$  is a subcategory of  $\mathcal{T}\text{-Cat}$ ).



## Theorem

For  $\varphi : X \dashv\dashv Y$  (i.e.  $\varphi : TX \times Y \longrightarrow V$ ), TFAE:

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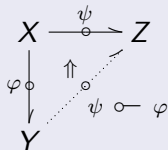
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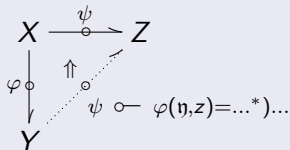
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$$*) \quad \bigwedge_{\substack{\mathfrak{W} \in \mathcal{T}(TX \times PX) \\ \mathfrak{W} \mapsto T \lceil \varphi \rceil (\eta)}} \text{hom}(\xi \cdot T \text{ev}(\mathfrak{W}), \psi(m_X \cdot T \pi_1(\mathfrak{W}), z))$$

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 X & & \psi(x) = [\mathcal{T}y(x), \psi]
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# Cocompleteness

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## Colimits

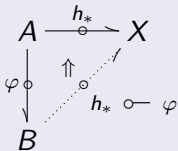
$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ \downarrow \varphi & & \\ B & & \end{array}$$

# Cocompleteness

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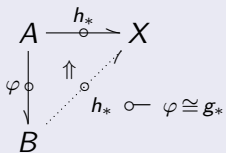
$$\begin{array}{ccc} A & \xrightarrow{h_*} & X \\ \downarrow \varphi & & \\ B & & \end{array}$$

## Colimits



# Cocompleteness

## Colimits



then  $g \cong \operatorname{colim}(\varphi, h)$ .

# Cocompleteness

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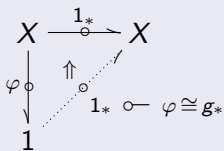
$$\begin{array}{ccc} X & \xrightarrow{1_*} & X \\ \varphi \downarrow & \nearrow & \\ B & \xrightarrow{1_*} & \varphi \cong g_* \end{array}$$

The diagram shows a commutative square. The top horizontal arrow is labeled  $1_*$  and has a small circle on it. The left vertical arrow is labeled  $\varphi$  and has a small circle on it. The bottom horizontal arrow is labeled  $1_*$  and has a small circle on it. The right vertical arrow is labeled  $\varphi \cong g_*$  and has a small circle on it. A dashed arrow points from  $B$  to the top-right  $X$ . A dashed arrow points from the top-left  $X$  to the top-right  $X$ . A dashed arrow points from the top-left  $X$  to the bottom-right  $\varphi \cong g_*$ . A dashed arrow points from the bottom-left  $B$  to the bottom-right  $\varphi \cong g_*$ . A dashed arrow points from the bottom-left  $B$  to the top-right  $X$ . A dashed arrow points from the top-left  $X$  to the bottom-right  $\varphi \cong g_*$ . A dashed arrow points from the bottom-left  $B$  to the top-right  $X$ .

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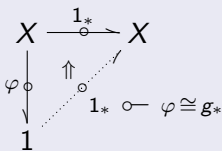
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(Not enough in Top!!)

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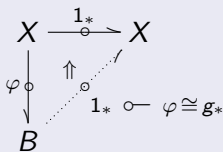
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# Cocompleteness

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## Theorem

- (i)  $X$  is cocomplete iff
- (ii)  $y_X : X \rightarrow PX$  has a left adjoint  $\text{Sup}_X : PX \rightarrow X$  iff
- (iii)  $X$  is injective w.r.t. embeddings.

## Theorem (under $T1 = 1$ )

- (i)  $X$  is 1-cocomplete iff
- (ii)  $y_X : X \rightarrow PX$  has a left adjoint in  $V\text{-Cat}$  iff
- (iii)  $X$  is complete and  $e_X : X \rightarrow MX$  preserves limits (in  $V\text{-Cat}$ ).

As expected:

We have an adjunction

$$\mathcal{T}\text{-Cat} \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} \mathcal{T}\text{-CD}^{\text{op}}$$

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which restricts to an equivalence

$$\begin{array}{ccc} \mathcal{T}\text{-Cat}_{\text{cc}} & \begin{array}{c} \xrightarrow{\quad} \\ \cong \\ \xleftarrow{\quad} \end{array} & \mathcal{T}\text{-Al}^{\text{op}} \\ \downarrow & & \downarrow \\ \mathcal{T}\text{-Dist} & \begin{array}{c} \xrightarrow{\quad} \\ \cong \\ \xleftarrow{\quad} \end{array} & \mathcal{T}\text{-Al}_{\text{cocont}}^{\text{op}} \end{array}$$

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 & \curvearrowright & \\
 & \perp & \\
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in Top and  $FL$  is (cd) via  $\mathcal{O}(FL) \rightarrow L$  in Frm.

# Everything is relative...

Let  $\Phi$  be a (non-full) subcategory of  $\mathcal{T}$ -Dist with

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 \\
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 \end{array} & & 
 \begin{array}{ccc}
 L \dashv\vdash A (\dashv\vdash L \rightrightarrows \Phi L) & & \\
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More general, we can chose  $\Phi =$  all those distributors  $\varphi : X \multimap Y$  where

$$\varphi \circ - : \mathbf{Dist}(1, X) \longrightarrow \mathbf{Dist}(1, Y)$$

preserves chosen limits.

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