Completely distributive spaces

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$$\begin{tikzcd}\mathsf{Ord} \arrow[r, shift left, "\mathsf{hom}(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left, "\mathsf{hom}(-{,}2)"]\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{Ord} \arrow[r, shift left, "\mathsf{hom}(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left, "\mathsf{hom}(-{,}2)"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{Ord} \arrow[r, shift left=1.5, "\mathsf{hom}(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left=1.5, "\mathsf{hom}(-{,}2)"]
\end{tikzcd}$$

$$X \longmapsto 2^{X^{\mathrm{op}}}$$

$$L \longmapsto [L, 2]^{\mathrm{op}}$$

$$\begin{tikzcd}
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\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & 2^{X^{\mathrm{op}}} \\
Y \arrow[r, mapsto] & 2^{Y^{\mathrm{op}}} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
L \arrow[r, mapsto] \arrow[d, "f"'] & {[L, 2]^{\mathrm{op}}} \\
M \arrow[r, mapsto] & {[M, 2]^{\mathrm{op}}} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{Ord} \arrow[r, shift left=2, "\mathsf{hom}(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left=2, "\mathsf{hom}(-{,}2)"]
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\end{tikzcd}$$

ccd-lattice:

$$\begin{tikzcd}[column sep=large]
L \arrow[r, bend left=45, "\ulcorner \ll \urcorner"] \arrow[r, bend right=45, "y_L"'] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, phantom, shift right=2.5, "\perp"] & PL \arrow[l, "\mathsf{Sup}" description]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{Ord} \arrow[r, shift left=1.5, "\mathrm{hom}(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left=1.5, "\mathrm{hom}(-{,}2)"]
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X \arrow[r, mapsto] \arrow[d, "f"'] & 2^{X^{\mathrm{op}}} \\
Y \arrow[r, mapsto] & 2^{Y^{\mathrm{op}}} \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, mapsto] \arrow[d, "f"', "f \vdash g"] & {[L,2]^{\mathrm{op}} \cong A} \\
M \arrow[r, mapsto] & {[M,2]^{\mathrm{op}} \cong B} \arrow[u, "g_0"']
\end{tikzcd}$$

ccd-lattice:

$$\begin{tikzcd}
A \arrow[r, "i"] & L \arrow[rr, bend left, "\ulcorner \ll \urcorner"] \arrow[rr, bend right, "y_L"'] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[rr, phantom, shift right=2, "\perp"] & & PL \arrow[ll, "\mathsf{Sup}" description]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{Ord} \arrow[r, shift left, "\mathsf{hom}(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left, "\mathsf{hom}(-{,}2)"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & 2^{X^{\mathrm{op}}} \\
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\end{tikzcd}$$

Distributor:

$$\varphi : X \mathrel{-\!\!\circ\!\!\rightarrow} Y \Leftrightarrow \varphi : X^{\mathrm{op}} \times Y \longrightarrow 2 \text{ monotone} \Leftrightarrow b \cdot \varphi = \varphi = \varphi \cdot a.$$

$$\begin{tikzcd}
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Distributor:

$\varphi : X \mathrel{-\!\!\!\circ\!\!\!\rightarrow} Y \Leftrightarrow \varphi : X^{\mathrm{op}} \times Y \longrightarrow 2$ monotone $\Leftrightarrow b \cdot \varphi = \varphi = \varphi \cdot a$.

For $f : X \longrightarrow Y$: $(f_* : X \mathrel{-\!\!\!\circ\!\!\!\rightarrow} Y) \dashv (f^* : Y \mathrel{-\!\!\!\circ\!\!\!\rightarrow} X)$.

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{Ord} \arrow[r, shift left=1.5ex, "\hom(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left=1.5ex, "\hom(-{,}2)"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & 2^{X^{\mathrm{op}}} \arrow[r, phantom, "\cong"] & \mathrm{Dist}(X,1) \arrow[d, dotted, bend right=60, "\vdash"] \arrow[d, dotted, bend left=60, "-\cdot f^{*}"] \\
Y \arrow[r, mapsto] & 2^{Y^{\mathrm{op}}} \arrow[r, phantom, "\cong"] & \mathrm{Dist}(Y,1) \arrow[u, "-\cdot f_{*} \vdash"']
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, mapsto] \arrow[d, "f"', "f \vdash g"] & {[L,2]^{\mathrm{op}}} \arrow[r, phantom, "\cong"] & A \\
M \arrow[r, mapsto] & {[M,2]^{\mathrm{op}}} \arrow[r, phantom, "\cong"] & B \arrow[u, "g_0"']
\end{tikzcd}$$

Distributor:

$\varphi : X \mathrel{-\mkern-6mu\circ\mkern-6mu\rightarrow} Y \Leftrightarrow \varphi : X^{\mathrm{op}} \times Y \longrightarrow 2$ monotone $\Leftrightarrow b \cdot \varphi = \varphi = \varphi \cdot a.$

For $f : X \longrightarrow Y$: $(f_* : X \mathrel{-\mkern-6mu\circ\mkern-6mu\rightarrow} Y) \dashv (f^* : Y \mathrel{-\mkern-6mu\circ\mkern-6mu\rightarrow} X)$.

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{Ord} \arrow[r, shift left=2, "\mathrm{hom}(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left=2, "\mathrm{hom}(-{,}2)"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & 2^{X^{\mathrm{op}}} \cong \mathrm{Dist}(X,1) \arrow[d, dotted, bend right=60, ""{name=A}] \arrow[d, dotted, bend left=60, ""{name=B}] \\
Y \arrow[r, mapsto] & 2^{Y^{\mathrm{op}}} \cong \mathrm{Dist}(Y,1) \arrow[u, "-\cdot f_*"'{name=C}]
\arrow[from=A, to=C, phantom, "\vdash"] \arrow[from=C, to=B, phantom, "\vdash -\cdot f^*"]
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, mapsto] \arrow[d, "f"', "f \vdash g"] & {[L,2]}^{\mathrm{op}} \cong A \\
M \arrow[r, mapsto] & {[M,2]}^{\mathrm{op}} \cong B \arrow[u, "g_0"']
\end{tikzcd}$$

About the units:

For $PX := \mathrm{Dist}(X,1)$: $\begin{tikzcd} X \arrow[r, "y_X"] & PX \arrow[rr, bend right, "y_{PX}"'] & & PPX \end{tikzcd}$

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{Ord} \arrow[r, shift left, "\mathrm{hom}(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left, "\mathrm{hom}(-{,}2)"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & 2^{X^{\mathrm{op}}} \cong \mathrm{Dist}(X,1) \arrow[d, dotted, bend right=50] \arrow[d, dotted, bend left=50, "-\cdot f^*"] \arrow[d, phantom, shift right=4ex, "\vdash"] \arrow[d, phantom, shift left=3ex, "\vdash"] \\
Y \arrow[r, mapsto] & 2^{Y^{\mathrm{op}}} \cong \mathrm{Dist}(Y,1) \arrow[u, "-\cdot f_*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, mapsto] \arrow[d, "f"', "f \dashv g"] & {[L,2]^{\mathrm{op}}} \cong A \\
M \arrow[r, mapsto] & {[M,2]^{\mathrm{op}}} \cong B \arrow[u, "g_0"']
\end{tikzcd}$$

About the units:

For $PX := \mathrm{Dist}(X,1)$: $\begin{tikzcd} X \arrow[r, "y_X"] & PX \arrow[r, bend right, "y_{PX}"'] \arrow[r, phantom, shift right=1ex, "\perp"] & PPX \arrow[l, "(-)\cdot y_{X*}"'] \end{tikzcd}$

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{Ord} \arrow[r, shift left=2, "\hom(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left=2, "\hom(-{,}2)"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & 2^{X^{\mathrm{op}}} \cong \mathrm{Dist}(X,1) \arrow[d, dotted, bend right=50, "\vdash"'] \arrow[d, dotted, bend left=50, "\vdash -\cdot f^*"] \\
Y \arrow[r, mapsto] & 2^{Y^{\mathrm{op}}} \cong \mathrm{Dist}(Y,1) \arrow[u, "-\cdot f_*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, mapsto] \arrow[d, "f"', "f \vdash g"] & {[L,2]^{\mathrm{op}}} \cong A \\
M \arrow[r, mapsto] & {[M,2]^{\mathrm{op}}} \cong B \arrow[u, "g_0"']
\end{tikzcd}$$

About the units:

For $PX := \mathrm{Dist}(X,1)$: $$\begin{tikzcd}
X \arrow[r, "y_X"] & PX \arrow[r, bend left=40, "(-)\cdot y_X^*"] \arrow[r, bend right=40, "y_{PX}"'] & PPX \arrow[l, "(-)\cdot y_{X*}", "\perp"', "\perp"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{Ord} \arrow[r, shift left, "\mathsf{hom}(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left, "\mathsf{hom}(-{,}2)"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & 2^{X^{\mathrm{op}}} \cong \mathrm{Dist}(X,1) \arrow[d, dotted, bend right, "\vdash"'] \arrow[d, dotted, bend left] \\
Y \arrow[r, mapsto] & 2^{Y^{\mathrm{op}}} \cong \mathrm{Dist}(Y,1) \arrow[u, "-\cdot f_* \vdash -\cdot f^*"']
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About the units:

For $PX := \mathrm{Dist}(X,1)$: $$\begin{tikzcd}
X \arrow[r, "y_X"] & PX \arrow[r, bend left, "(-)\cdot y_X^*"] \arrow[r, bend right, "y_{PX}"'] & PPX \arrow[l, "(-)\cdot y_{X*}"' , "\perp", "\perp"']
\end{tikzcd}$$

For L in CCD: $L \xrightarrow{\varepsilon_L} PA$, $x \longmapsto \{z \in A \mid z \leq x\}$;

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{Ord} \arrow[r, shift left, "\mathsf{hom}(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left, "\mathsf{hom}(-{,}2)"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & 2^{X^{\mathrm{op}}} \cong \mathsf{Dist}(X,1) \arrow[d, dotted, bend right, "\vdash"'] \arrow[d, dotted, bend left, "-\cdot f^*"] \\
Y \arrow[r, mapsto] & 2^{Y^{\mathrm{op}}} \cong \mathsf{Dist}(Y,1) \arrow[u, "-\cdot f_* \vdash"']
\end{tikzcd}$$

$$\begin{tikzcd}
L \arrow[r, mapsto] \arrow[d, "f"', "f \vdash g"] & {[L,2]^{\mathrm{op}}} \cong A \\
M \arrow[r, mapsto] & {[M,2]^{\mathrm{op}}} \cong B \arrow[u, "g_0"']
\end{tikzcd}$$

About the units:

For $PX := \mathsf{Dist}(X,1)$: $$\begin{tikzcd}
X \arrow[r, "y_X"] & PX \arrow[r, bend left, "(-)\cdot y_X^*"] \arrow[r, bend right, "y_{PX}"'] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & PPX \arrow[l, "(-)\cdot y_{X*}"']
\end{tikzcd}$$

For L in CCD: $L \xrightarrow{\varepsilon_L} PA$, $x \longmapsto x^* \cdot i_*$;

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{Ord} \arrow[r, shift left, "\hom(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left, "\hom(-{,}2)"]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & 2^{X^{\mathrm{op}}} \cong \mathrm{Dist}(X,1) \arrow[d, dotted, bend right=60, "\vdash"'] \arrow[d, dotted, bend left=60, "\vdash -\cdot f^*"] \\
Y \arrow[r, mapsto] & 2^{Y^{\mathrm{op}}} \cong \mathrm{Dist}(Y,1) \arrow[u, "-\cdot f_*"]
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\end{tikzcd}$$

About the units:

For $PX := \mathrm{Dist}(X,1)$: $$\begin{tikzcd}
X \arrow[r, "y_X"] & PX \arrow[r, bend left, "(-)\cdot y_X^*"] \arrow[r, bend right, "y_{PX}"'] & PPX \arrow[l, "(-)\cdot y_{X*}"' description] \arrow[l, phantom, shift right=3, "\perp"] \arrow[l, phantom, shift left=3, "\perp"]
\end{tikzcd}$$

For L in CCD: $L \xrightarrow{\varepsilon_L} PA,\ x \longmapsto x^* \cdot i_*;\ \ \varepsilon_L \cdot S = 1,\ S \cdot \varepsilon_L \leq 1.$

$$\begin{tikzcd}
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$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & 2^{X^{\mathrm{op}}} \cong \mathsf{Dist}(X,1) \arrow[d, dotted, bend right, "\vdash"] \arrow[d, dotted, bend left, "\vdash -\cdot f^*"] \\
Y \arrow[r, mapsto] & 2^{Y^{\mathrm{op}}} \cong \mathsf{Dist}(Y,1) \arrow[u, "-\cdot f_*"']
\end{tikzcd}$$

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\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, shift left] \arrow[r, phantom, "\cong"] & \mathsf{Tal}^{\mathrm{op}} \arrow[l, shift left]
\end{tikzcd}$$

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X \arrow[r, mapsto] \arrow[d, "f"'] & 2^{X^{\mathrm{op}}} \cong \mathsf{Dist}(X,1) \arrow[d, dotted, bend right, ""{name=A}] \arrow[d, dotted, bend left, "-\cdot f^*", ""'{name=C}] \\
Y \arrow[r, mapsto] & 2^{Y^{\mathrm{op}}} \cong \mathsf{Dist}(Y,1) \arrow[u, "-\cdot f_*"', ""{name=B}] \arrow[from=A, to=B, phantom, "\vdash"] \arrow[from=B, to=C, phantom, "\vdash"]
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L \arrow[r, mapsto] \arrow[d, "f"', "f\vdash g"] & {[L,2]^{\mathrm{op}} \cong A} \\
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$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, shift left] \arrow[r, phantom, "\cong"] \arrow[d] & \mathsf{Tal}^{\mathrm{op}} \arrow[l, shift left] \arrow[d] \\
\mathsf{Dist} \arrow[r, shift left] \arrow[r, phantom, "\cong"] & \mathsf{Tal}^{\mathrm{op}}_{\mathrm{sup}} \arrow[l, shift left]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{Ord} \arrow[r, "(-)^{\mathrm{op}}"] & \mathsf{Ord} \arrow[r, shift left=1.5, "\mathsf{hom}(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{CCD}^{\mathrm{op}} \arrow[l, shift left=1.5, "\mathsf{hom}(-{,}2)"]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large, row sep=huge]
X \arrow[r, mapsto] \arrow[d, "f"'] & {2^{X^{\mathrm{op}}} \cong \mathsf{Dist}(X,1)} \arrow[d, dotted, bend right=50, ""{name=A}] \arrow[d, dotted, bend left=50, "\vdash -\cdot f^*"] \\
Y \arrow[r, mapsto] & {2^{Y^{\mathrm{op}}} \cong \mathsf{Dist}(Y,1)} \arrow[u, "-\cdot f_*"', "\vdash"]
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$$\begin{tikzcd}[column sep=large, row sep=huge]
L \arrow[r, mapsto] \arrow[d, "f"', "f^*=g_*"] & {[L,2]^{\mathrm{op}} \cong A} \\
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\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathsf{Ord} \arrow[r, shift left=1.5] \arrow[r, phantom, "\cong"] \arrow[d] & \mathsf{Tal}^{\mathrm{op}} \arrow[l, shift left=1.5] \arrow[d] \\
\mathsf{Dist} \arrow[r, shift left=1.5] \arrow[r, phantom, "\cong"] & \mathsf{Tal}^{\mathrm{op}}_{\mathrm{sup}} \arrow[l, shift left=1.5]
\end{tikzcd}$$

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X \arrow[r, mapsto] \arrow[d, "f"'] & 2^{X^{\mathrm{op}}} \cong \mathsf{Dist}(X,1) \arrow[d, dotted, bend right, "\vdash"] \arrow[d, dotted, bend left, "\vdash -\cdot f^*"] \\
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\end{tikzcd}$$

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\mathsf{Ord} \arrow[r, shift left, "\cong"'] \arrow[d] & \mathsf{Tal}^{\mathrm{op}} \arrow[l, shift left] \arrow[d] \\
\mathsf{Dist} \arrow[r, shift left, "\cong"'] \arrow[d] & \mathsf{Tal}^{\mathrm{op}}_{\mathrm{sup}} \arrow[l, shift left] \arrow[d] \\
\mathsf{kar}(\mathsf{Rel}) \arrow[r, shift left, "\cong"'] & \mathsf{CCD}^{\mathrm{op}}_{\mathrm{sup}} \arrow[l, shift left]
\end{tikzcd}$$

Topological spaces

M. Barr (1970):

Topological space = set X with $a : UX \times X \longrightarrow 2$ such that

$$\dot{x} \longrightarrow x, \qquad (\mathfrak{X} \longrightarrow \mathfrak{x} \,\&\, \mathfrak{x} \longrightarrow x) \Rightarrow (m_X(\mathfrak{X}) \longrightarrow x).$$

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From here we have two choices:

- $X \longmapsto 2^X$ leads to $$\begin{tikzcd} \mathsf{Top} \arrow[r, shift left, "\mathsf{hom}(-,2)"] \arrow[r, phantom, "\perp"] & \mathsf{Frm}^{\mathrm{op}} \arrow[l, shift left, "\mathsf{hom}(-,2)"] \end{tikzcd}$$

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From here we have two choices:

- $X \longmapsto 2^X$ leads to $$\begin{tikzcd} \mathsf{Top} \arrow[r, shift left, "\hom(-{,}2)"] \arrow[r, phantom, "\perp"] & \mathsf{Frm}^{\mathrm{op}} \arrow[l, shift left, "\hom(-{,}2)"] \end{tikzcd}$$
- $X \longmapsto 2^{UX}$ leads to . . . ?. . .

The dictionary

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relation $r : X \times Y \longrightarrow 2$	

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The dictionary

$\mathsf{V} = (\mathsf{V}, \otimes, k)$	$\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi : T\mathsf{V} \to \mathsf{V})$
V-relation $r : X \times Y \longrightarrow \mathsf{V}$	$\mathcal{T}$-relation $r : TX \times Y \longrightarrow \mathsf{V}$

The dictionary

$\mathsf{V} = (\mathsf{V}, \otimes, k)$	$\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi : T\mathsf{V} \to \mathsf{V})$
$r : X \relbar\!\!\!\mapsto Y$	$\mathcal{T}$-relation $r : TX \times Y \longrightarrow \mathsf{V}$

The dictionary

$\mathsf{V} = (\mathsf{V}, \otimes, k)$	$\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi : TV \to \mathsf{V})$
$r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$	$r : X \relbar\joinrel\mapstochar\joinrel\rightharpoondown Y$

The dictionary

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$r : X \mathrel{\longrightarrow\kern-1.6em\mapstochar\kern1.6em} Y$	$r : X \mathrel{\relbar\kern-0.8em\mapstochar\kern0.2em\rightharpoonup} Y$
relational composition	

The dictionary

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$r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ relational composition	$r : X \relbar\joinrel\mapstochar\joinrel\rightharpoondown Y$ Kleisli composition

Kleisli composition:

- $(TX \overset{r}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Y, TY \overset{s}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Z) \longmapsto (TX \overset{m_X^\circ}{\relbar\joinrel\mapstochar\joinrel\rightarrow} TTX \overset{Tr}{\relbar\joinrel\mapstochar\joinrel\rightarrow} TY \overset{s}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Z)$
- One has $e_Y^\circ \circ r \geq r$ and $r \circ e_X^\circ = r$.

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V-category: $a : X \relbar\joinrel\mapstochar\joinrel\rightarrow X$ with $1_X \leq a$ and $a \cdot a \leq a$.

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X^{op}	??

Recall:

- $$\begin{aligned}\varphi:X\relbar\joinrel\circ\joinrel\rightarrow Y &\iff \varphi:X^{\mathrm{op}}\otimes Y\longrightarrow\mathsf{V}\text{ is a }\mathsf{V}\text{-functor}\\ &\iff \ulcorner\varphi\urcorner:Y\longrightarrow\mathsf{V}^{X^{\mathrm{op}}}\text{ is a }\mathsf{V}\text{-functor.}\end{aligned}$$

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Yoneda	??
Supremum (weighted colimit)	??

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The dual category

The dual category

- $\mathbb{T}$ extends to a monad on V-Cat via

$$(X, a : X \relbar\!\!\!\mapsto X) \longmapsto (TX, Ta : TX \relbar\!\!\!\mapsto TX).$$

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- We have a functor

$$\begin{aligned} K : \text{V-Cat}^{\mathbb{T}} &\longrightarrow \mathcal{T}\text{-Cat} \\ (X, a_0, \alpha) &\longmapsto (X, a_0 \cdot \alpha : TV \relbar\joinrel\mapstochar\joinrel\rightarrow V) \end{aligned}$$

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$$\begin{aligned} M : \mathcal{T}\text{-}\mathsf{Cat} &\longrightarrow \mathsf{V}\text{-}\mathsf{Cat}^{\mathbb{T}}. \\ (X, a) &\longmapsto (TX, Ta \cdot m_X, m_X) \end{aligned}$$

Here: $TX \xrightarrow{m_X^\circ} TTX \xrightarrow{Ta} TX$

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The dual category

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- Hence $\mathbb{T}$ extends to a monad on $\mathcal{T}$-Cat induced by $M \dashv K$ which is of KZ-type.

The dual category

Hence we define:

$$\begin{tikzcd}
\mathcal{T}\text{-Cat} \arrow[r, dotted, "(-)^{\mathrm{op}}"] \arrow[d, "M"'] & \mathcal{T}\text{-Cat} \\
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- Every $K(X)$ is $\otimes$-exponentiable in $\mathcal{T}$-Cat.

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More Properties:

- Every $K(X)$ is $\otimes$-exponentiable in $\mathcal{T}$-Cat.
- $\mathbb{T}$ is a Hopf-monad on V-Cat making V-Cat$^{\mathbb{T}}$ monoidal.
- K : V-Cat$^{\mathbb{T}} \longrightarrow \mathcal{T}$-Cat is strict monoidal.
- V-Cat$^{\mathbb{T}} \cong \mathcal{T}$-Cat$^{\mathbb{T}}$; ($\mathcal{T}$-Cat$^{\mathbb{T}}$ is a subcategory of $\mathcal{T}$-Cat).

Yoneda

Yoneda

Theorem

For $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ *(i.e.* $\varphi : TX \times Y \longrightarrow \mathsf{V}$*), TFAE:*

- (i) $\varphi : X \mathrel{-\!\!\circ\!\!\rightharpoonup} Y$ *is a* $\mathcal{T}$*-distributor.*
- (ii) $\varphi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathsf{V}$ *is a* $\mathcal{T}$*-functor.*
- (iii) $\ulcorner\varphi\urcorner : X \longrightarrow V^{X^{\mathrm{op}}} =: PX$ *is a* $\mathcal{T}$*-functor.*

Yoneda

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Hence, from $a : X \relbar\joinrel\circ\joinrel\rightharpoonup X$ *we get* $y_X := \ulcorner a \urcorner : X \longrightarrow PX$.

Yoneda

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Recall that, for a V-distributor $\varphi : X \relbar\!\!\circ\!\!\rightarrow Y$

$$- \cdot \varphi : \mathsf{V}\text{-}\mathrm{Dist}(Y, Z) \longrightarrow \mathsf{V}\text{-}\mathrm{Dist}(X, Z) \text{ is left adjoint.}$$

Yoneda

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Recall that, for a V-distributor $\varphi : X \mathrel{-\!\!\circ\!\!\rightarrow} Y$

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$$\begin{tikzcd}
X \arrow[r, "\psi", "\circ" marking] \arrow[d, "\varphi"', "\circ" marking] & Z \\
Y \arrow[ur, dotted, "\circ" marking, "\Uparrow"] &
\end{tikzcd}$$

$\psi \bullet\!\!- \varphi$

Yoneda

Theorem

For $\varphi : X \mathrel{-\!\!\!+\!\!\!\rightharpoonup} Y$ *(i.e.* $\varphi : TX \times Y \longrightarrow \mathsf{V}$*), TFAE:*

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Hence, from $a : X \mathrel{-\!\!\!\circ\!\!\!\rightharpoonup} X$ *we get* $y_X := \ulcorner a\urcorner : X \longrightarrow PX$.

Recall that, for a V-distributor $\varphi : X \mathrel{-\!\!\!\circ\!\!\!\rightarrow} Y$

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$$\begin{tikzcd}
X \arrow[r, "\psi", "\circ" marking] \arrow[d, "\varphi"', "\circ" marking] \arrow[dr, phantom, "\Uparrow"] & Z \\
Y \arrow[ru, dotted, "\circ" marking] &
\end{tikzcd}$$

$$\psi \bullet\!\!- \varphi(y,z) = \bigwedge_x \hom(\varphi(x,y), \psi(x,z))$$

Yoneda

Theorem

For $\varphi : X \relbar\joinrel\mapstochar\joinrel\rightharpoondown Y$ *(i.e.* $\varphi : TX \times Y \longrightarrow \mathsf{V}$*), TFAE:*

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- (iii) $\ulcorner\varphi\urcorner : X \longrightarrow V^{X^{\mathrm{op}}} =: PX$ *is a* $\mathfrak{T}$*-functor.*

Hence, from $a : X \relbar\joinrel\circ\joinrel\rightharpoondown X$ *we get* $y_X := \ulcorner a\urcorner : X \longrightarrow PX$.

Recall that, for a V-distributor $\varphi : X \relbar\joinrel\circ\joinrel\rightharpoondown Y$

$$- \cdot \varphi : \mathsf{V}\text{-}\mathsf{Dist}(Y, Z) \longrightarrow \mathsf{V}\text{-}\mathsf{Dist}(X, Z) \text{ is left adjoint.}$$

$$\begin{tikzcd}
X \arrow[r, "\psi", "\circ"{marking}] \arrow[d, "\varphi"', "\circ"{marking}] & Z \\
Y \arrow[ur, dotted, "\circ"{marking}, "\Uparrow"] &
\end{tikzcd}$$

$$\psi \;\bullet\!\!\relbar\; \varphi(y,z) = \bigwedge_{x} \hom(\varphi(x,y), \psi(x,z)) = [\ulcorner\varphi\urcorner(y), \ulcorner\psi\urcorner(z)]$$

Here: $\ulcorner\varphi\urcorner : Y \longrightarrow PX$, $\ulcorner\psi\urcorner : Z \longrightarrow PX$.

Yoneda

Theorem

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Hence, from $a : X \mathrel{-\!\!\circ\!\!\rightharpoondown} X$ *we get* $y_X := \ulcorner a\urcorner : X \longrightarrow PX$.

Simularly, for a $\mathcal{T}$-distributor $\varphi : X \mathrel{-\!\!\circ\!\!\rightharpoondown} Y$,

$$- \circ \varphi : \mathcal{T}\text{-Dist}(Y, Z) \longrightarrow \mathcal{T}\text{-Dist}(X, Z) \text{ is left adjoint.}$$

Yoneda

Theorem

For $\varphi : X \relbar\!\!\mapstochar\!\!\rightharpoonup Y$ *(i.e.* $\varphi : TX \times Y \longrightarrow \mathsf{V}$*), TFAE:*

1. $\varphi : X \relbar\!\!\circ\!\!\rightharpoonup Y$ *is a* $\mathcal{T}$*-distributor.*
2. $\varphi : X^{\mathrm{op}} \otimes Y \longrightarrow \mathsf{V}$ *is a* $\mathcal{T}$*-functor.*
3. $\ulcorner \varphi \urcorner : X \longrightarrow V^{X^{\mathrm{op}}} =: PX$ *is a* $\mathcal{T}$*-functor.*

Hence, from $a : X \relbar\!\!\circ\!\!\rightharpoonup X$ *we get* $y_X := \ulcorner a \urcorner : X \longrightarrow PX$.

Simularly, for a $\mathcal{T}$-distributor $\varphi : X \relbar\!\!\circ\!\!\rightharpoonup Y$,

$$- \circ \varphi : \mathcal{T}\text{-Dist}(Y, Z) \longrightarrow \mathcal{T}\text{-Dist}(X, Z) \text{ is left adjoint.}$$

$$\begin{tikzcd}
X \arrow[r, "\psi"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\Uparrow"] & Z \\
Y \arrow[ur, dotted, "\psi \mathrel{\circ\!\!-} \varphi"'] &
\end{tikzcd}$$

Yoneda

Theorem

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X \arrow[r, "\psi", "\circ" marking] \arrow[d, "\varphi"', "\circ" marking] & Z \\
Y \arrow[ur, dotted, "\circ" marking, "\psi \mathrel{\circ\!\!-} \varphi(\mathfrak{y},z)=\ldots^{*)}\ldots"'] \arrow[ur, phantom, shift left=3ex, "\Uparrow"] &
\end{tikzcd}$$

$^{*)}$ $$\bigwedge_{\substack{\mathfrak{W}\in T(TX\times PX)\\ \mathfrak{W}\longmapsto T\ulcorner\varphi\urcorner(\mathfrak{y})}} \hom(\xi \cdot T\,\mathrm{ev}(\mathfrak{W}), \psi(m_X \cdot T\pi_1(\mathfrak{W}), z))$$

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\end{tikzcd}$$

$$\psi \mathrel{\circ\!\!-} \varphi(\mathfrak{y},z)=\ldots{}^{*})\ldots=[\![T\ulcorner\varphi\urcorner(\mathfrak{y}),\ulcorner\psi\urcorner(z)]\!]$$

Here: $\ulcorner\varphi\urcorner : Y \longrightarrow PX$, $\ulcorner\psi\urcorner : Z \longrightarrow PX$.

Yoneda

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$$\begin{tikzcd}
X \arrow[r, "\psi", "\circ"{marking}] \arrow[d, "a=\varphi"', "\circ"{marking}] & 1 \\
X \arrow[ur, dotted, "\circ"{marking}, "\Uparrow"]
\end{tikzcd}$$

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Yoneda

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Simularly, for a $\mathcal{T}$-distributor $\varphi : X \mathrel{-\!\!\circ\!\!\rightharpoondown} Y$,

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$$\begin{tikzcd}
X \arrow[r, harpoon, "\psi", "\circ"{marking}] \arrow[d, harpoon, "a=\varphi"', "\circ"{marking}] & 1 \\
X \arrow[ur, dotted, harpoon, "\Uparrow", "\circ"{marking}, "\psi(\mathfrak{x})=[\![T y(\mathfrak{x}),\psi]\!]"'] &
\end{tikzcd}$$

Cocompleteness

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Colimits

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "\varphi"', "\circ" marking] & X \\
B &
\end{tikzcd}$$

Cocompleteness

Colimits

$$\begin{tikzcd}
A \arrow[r, "h_*", "\circ" marking] \arrow[d, "\varphi"', "\circ" marking] & X \\
B &
\end{tikzcd}$$

Cocompleteness

Colimits

$$\begin{tikzcd}
A \arrow[r, "h_*"] \arrow[r, "\circ" marking] \arrow[d, "\varphi"'] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\Uparrow"] & X \\
B \arrow[ur, dotted, "h_* \circ\!\!-\, \varphi"'] \arrow[ur, dotted, "\circ" marking] &
\end{tikzcd}$$

Cocompleteness

Colimits

$$\begin{tikzcd} A \arrow[r, "h_*", "\circ" marking] \arrow[d, "\varphi"', "\circ" marking] \arrow[dr, phantom, "\Uparrow"] & X \\ B \arrow[ur, dotted, "\circ" marking, "h_* \circ\!\!- \varphi \cong g_*"'] & \end{tikzcd}$$

then $g \cong \mathrm{colim}(\varphi, h)$.

Cocompleteness

Colimits

$$\begin{tikzcd}
X \arrow[r, "1_*", "\circ" marking] \arrow[d, "\varphi"', "\circ" marking] & X \\
B \arrow[ur, dotted, "\circ" marking, "1_* \circ\!\!- \varphi \cong g_*"'] \arrow[u, phantom, "\Uparrow", shift left=6ex]
\end{tikzcd}$$

then $g \cong \mathrm{colim}(\varphi, 1)$.

Cocompleteness

Colimits

$$\begin{tikzcd}
X \arrow[r, "1_*", "\circ" marking] \arrow[d, "\varphi"', "\circ" marking] \arrow[dr, phantom, "\Uparrow" near start] & X \\
1 \arrow[ru, dotted, "\circ" marking, "1_* \mathrel{\circ\!\!-} \varphi \cong g_*"'] &
\end{tikzcd}$$

then $g \cong \mathrm{Sup}(\varphi)$.

Cocompleteness

Colimits

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then $g \cong \mathrm{Sup}(\varphi)$.
(Not enough in Top!!)

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\end{tikzcd}$$

then $g \cong \mathrm{colim}(\varphi, 1)$.

Theorem

- (i) *X is cocomplete iff*
- (ii) *$\text{よ}_X : X \longrightarrow PX$ has a left adjoint $\mathrm{Sup}_X : PX \longrightarrow X$ iff*
- (iii) *X is injective w.r.t. embeddings.*

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Theorem

1. *X is cocomplete iff*
2. *$y_X : X \longrightarrow PX$ has a left adjoint* $\mathrm{Sup}_X : PX \longrightarrow X$ *iff*
3. *X is injective w.r.t. embeddings.*

Theorem (under $T1 = 1$)

1. *X is 1-cocomplete iff*
2. *$y_X : X \longrightarrow PX$ has a left adjoint in* V-Cat *iff*
3. *X is complete and $e_X : X \longrightarrow MX$ preserves limits (in* V-Cat*).*

As expected:

We have an adjunction

$$\begin{tikzcd} \mathcal{T}\text{-Cat} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{T}\text{-CD}^{\mathrm{op}} \arrow[l, shift left] \end{tikzcd}$$

As expected:

We have an adjunction

$$\begin{tikzcd}
\mathcal{T}\text{-}\mathsf{Cat} \arrow[r, shift left=1.5ex] \arrow[r, phantom, "\perp"] & \mathcal{T}\text{-}\mathsf{CD}^{\mathrm{op}} \arrow[l, shift left=1.5ex]
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, "f"'] & PX \cong \mathcal{T}\text{-}\mathsf{Dist}(X,1) \arrow[d, dotted, bend right=60, "\vdash"] \arrow[d, dotted, bend left=60, "-\cdot f^*"] & L \arrow[r, mapsto] \arrow[d, "f"', "f \vdash g"] & A\ (\rightarrowtail L \rightrightarrows PL) \\
Y \arrow[r, mapsto] & PY \cong \mathcal{T}\text{-}\mathsf{Dist}(Y,1) \arrow[u, "-\cdot f_* \vdash"'] & M \arrow[r, mapsto] & B\ (\rightarrowtail M \rightrightarrows PM) \arrow[u, "g_0"']
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which restricts to an equivalence

$$\begin{tikzcd} \mathcal{T}\text{-Cat}_{\mathrm{cc}} \arrow[r, shift left] \arrow[r, phantom, "\cong"] \arrow[d] & \mathcal{T}\text{-Al}^{\mathrm{op}} \arrow[l, shift left] \arrow[d] \\ \mathcal{T}\text{-Dist} \arrow[r, shift left] \arrow[r, phantom, "\cong"] & \mathcal{T}\text{-Al}^{\mathrm{op}}_{\mathrm{cocont}} \arrow[l, shift left] \end{tikzcd}$$

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- $2^{X^{\mathrm{op}}} \longrightarrow F(\mathcal{O}X)$, $\mathcal{A} \longmapsto (\bigcap \mathcal{A}) \cap \tau$ is a homeomorphism.

  Here $F(\mathcal{O}X)$ is the set of all filters of open subsets with $\{\mathfrak{f} \in F(\mathcal{O}X) \mid A \in \mathfrak{f}\}$ where $A \subseteq X$ is open as basic open sets.

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- $$\begin{tikzcd} \mathsf{Top} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathsf{CDTop}^{\mathrm{op}} \arrow[l, shift left] \end{tikzcd}$$ restricts to $$\begin{tikzcd} \mathsf{Sob} \arrow[r, shift left] \arrow[r, phantom, "\cong"] & \mathsf{TATop}^{\mathrm{op}} \arrow[l, shift left] \end{tikzcd}$$.

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- For each frame homomorphism $f : L \longrightarrow M$ one gets

$$\begin{tikzcd}
FL \arrow[r, "Ff" description] & FM \arrow[l, bend right, "\perp"'] \arrow[l, bend left, "\perp"]
\end{tikzcd}$$

in Top

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- For each frame homomorphism $f : L \longrightarrow M$ one gets

$$\begin{tikzcd} FL \arrow[r, "Ff" description] \arrow[r, phantom, shift left=2.5ex, "\perp"] \arrow[r, phantom, shift right=2.5ex, "\perp"] & FM \arrow[l, bend right=40] \arrow[l, bend left=40] \end{tikzcd}$$

in Top and FL is (cd) via $\mathcal{O}(FL) \longrightarrow L$ in Frm.

Everything is relative. . .

Let Φ be a (non-full) subcategory of $\mathcal{T}$-Dist with

- $f^* \in \Phi$, for every $\mathcal{T}$-functor f, and
- For all $\varphi : X \mathrel{\circ\!\!\rightharpoonup} Y$: $(\forall y \in Y \,.\, y^* \circ \varphi \in \Phi) \Rightarrow \varphi \in \Phi$.

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Then:

$$\begin{aligned} X \text{ is } \Phi\text{-cocomplete} &\iff \mathcal{y}_X : X \longrightarrow \Phi(X) \text{ has a left adjoint} \\ &\iff X \text{ is injective w.r.t. } \Phi\text{-dense embeddings.} \end{aligned}$$

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We obtain an adjunction

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$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & \Phi X \arrow[d, dotted, bend left] \\ Y \arrow[r, mapsto] & \Phi Y \arrow[u, "- \cdot f_* \vdash - \cdot f^*"'] \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, mapsto] \arrow[d, "f"', "f \dashv g"] & A \; (\rightarrowtail L \rightrightarrows \Phi L) \\ M \arrow[r, mapsto] & B \; (\rightarrowtail M \rightrightarrows \Phi M) \arrow[u, "g_0"'] \end{tikzcd}$$

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which restricts to

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“inhabited” distributors

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For $f : X \longrightarrow Y$ in Top:

$$f : X \longrightarrow Y \text{ is dense} \iff \forall y \in Y \, \exists \mathfrak{x} \in UX \,.\, Uf(\mathfrak{x}) \to y$$

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Then:

- $\Phi X \cong$ all proper filters on $\mathcal{O}X$.

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$$\varphi : X \mathrel{-\!\!\circ\!\!\rightharpoonup} Y \in \Phi \text{ whenever } \forall y \in Y \, \exists \mathfrak{x} \in UX \,.\, \mathfrak{x} \, \varphi \, y.$$

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More general, we can chose $\Phi =$ all those distributors $\varphi : X \mathrel{-\!\!\circ\!\!\rightharpoondown} Y$ where

$$\varphi \circ - : \mathsf{Dist}(1, X) \longrightarrow \mathsf{Dist}(1, Y)$$

preserves chosen limits.

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