Dualities for distributive spaces

Dirk Hofmann

Departamento de Matemática & CIDMA, Universidade de Aveiro, Portugal
http://www.mat.ua.pt/pessoais/dirk/
dirk@ua.pt

2011 Spring Southeastern AMS Sectional Meeting

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$

X ordered set: $x \leq x, \quad (x \leq y \leq z) \Rightarrow (x \leq z)$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$

X top. space: $\dot{x} \to x, \quad (\mathfrak{X} \to \mathfrak{x} \to x) \Rightarrow (m_X(\mathfrak{X}) \to x)$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$

X top. space: $\dot{x} \to x, \quad (\mathfrak{X} \to \mathfrak{x} \to x) \Rightarrow (m_X(\mathfrak{X}) \to x)$ where

$\dot{x} = \{A \subseteq X \mid x \in A\}$, $m_X(\mathfrak{X}) = \{A \subseteq X \mid \mathfrak{X} \in A^{\#}\}$

$(A^{\#} = \{\mathfrak{x} \in UX \mid A \in \mathfrak{x}\})$.

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset

$=$ morphism of type $X \to 2$

(Sierpiński space $2 = \{0, 1\}$ with $\{1\}$ closed)

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset

$=$ morphism of type $X \to 2$

(Sierpiński space $2 = \{0, 1\}$ with $\{1\}$ closed)

Example: $\uparrow x = \{y \in X \mid x \leq y\}$

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset

$=$ morphism of type $X \to 2$

(Sierpiński space $2 = \{0, 1\}$ with $\{1\}$ closed)

Example: $\uparrow\mathfrak{x} = \{x \in X \mid \mathfrak{x} \to x\}$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	

$=$ morphism of type $X^{\mathrm{op}} \to 2$

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	

$=$ morphism of type $X^{\mathrm{op}} \to 2$ resp. $X \to 2^{\mathrm{op}}$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	

$=$ morphism of type $X^{\mathrm{op}} \to 2$ resp. $X \to 2^{\mathrm{op}}$

Example: $\downarrow y = \{x \in X \mid x \leq y\}$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	

$=$ morphism of type $X^{\mathrm{op}} \to 2$ resp. $X \to 2^{\mathrm{op}}$

Example: $\downarrow x = \{\mathfrak{x} \in X \mid \mathfrak{x} \to x\} \subseteq UX$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	

Hence: $X^{\mathrm{op}} = (UX, \tau)$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	

Hence: $X^{\mathrm{op}} = (UX, \tau)$,

$\mathcal{A} \subseteq UX$ closed $\iff$ $\mathcal{A} = \{\mathfrak{x} \mid \mathfrak{x} \supseteq \mathfrak{f}\}$ for some filter of opens $\mathfrak{f} \subseteq \mathcal{O}X$.

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens

Hence: $X^{\mathrm{op}} = (UX, \tau)$,

$\mathcal{A} \subseteq UX$ closed $\iff$ $\mathcal{A} = \{\mathfrak{x} \mid \mathfrak{x} \supseteq \mathfrak{f}\}$ for some filter of opens $\mathfrak{f} \subseteq \mathcal{O}X$.

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point

$x \leq y$ whenever $\dot{x} \to y$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point

$x \leq y$ whenever $\dot{x} \to y \quad (\iff y \in \overline{\{x\}})$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point

Adjunction: for $f : X \to Y$ and $g : Y \to X$,

$f \dashv g \iff 1_X \leq g \cdot f \;\&\; f \cdot g \leq 1_Y \iff (Uf(\mathfrak{x}) \to y \Leftrightarrow \mathfrak{x} \to g(y))$

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point

Adjunction: for $f : X \to Y$ and $g : Y \to X$,

$f \dashv g \iff 1_X \leq g \cdot f \;\&\; f \cdot g \leq 1_Y \iff (Uf(\mathfrak{x}) \to y \Leftrightarrow \mathfrak{x} \to g(y))$

Fact: f left adjoint $\Rightarrow$ f preserves smallest limit points.

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	

In Ord: X cocomplete $\iff y_X : X \to 2^{X^{\mathrm{op}}}, x \mapsto \downarrow x$ has left adjoint.

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	

In Top: we wish to have a left adjoint of $y_X : X \to 2^{X^{\mathrm{op}}}$, $x \mapsto \downarrow x$.

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	

In Top: we wish to have a left adjoint of $y_X : X \to 2^{X^{\mathrm{op}}}$, $x \mapsto \downarrow x$.

Note: $\to$: $X^{\mathrm{op}} \times X \to 2$ is continuous, X^{op} exponentiable.

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	cocomplete

In Top: we wish to have a left adjoint of $y_X : X \to 2^{X^{\mathrm{op}}}$, $x \mapsto {\downarrow}x$.

Note: $\to : X^{\mathrm{op}} \times X \to 2$ is continuous, X^{op} exponentiable.

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	cocomplete
down-set monad $\mathbb{D}$	

$\mathbb{D} = (D, y, m)$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	cocomplete
down-set monad $\mathbb{D}$	

$\mathbb{D} = (D, y, m)$, where $DX = 2^{X^{\mathrm{op}}}$,

$y_X : X \to DX$, $x \mapsto \downarrow x$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	cocomplete
down-set monad $\mathbb{D}$	

$\mathbb{D} = (D, y, m)$, where $DX = 2^{X^{\mathrm{op}}}$,

$y_X : X \to DX$, $x \mapsto \downarrow x$, and

$m_X : DDX \to DX, \mathcal{A} \mapsto \bigcup \mathcal{A}$

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	cocomplete
down-set monad $\mathbb{D}$	

$\mathbb{D} = (D, y, m)$, where $DX = 2^{X^{\mathrm{op}}}$,

$y_X : X \to DX$, $x \mapsto {\downarrow}x$, and

$m_X : DDX \to DX, \mathcal{A} \mapsto \bigcup \mathcal{A} = \{x \in X \mid y_X(x) \in \mathcal{A}\}$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	cocomplete
down-set monad $\mathbb{D}$	

For Top: $X \mapsto 2^{X^{\mathrm{op}}}$

$y_X : X \to 2^{X^{\mathrm{op}}}$

$m_X(\Psi) = \{\mathfrak{x} \in UX \mid Uy_X(\mathfrak{x}) \in \Psi\}$

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	cocomplete
down-set monad $\mathbb{D}$	

For Top: $X \mapsto 2^{X^{\mathrm{op}}} \simeq FX$ with basic opens $A^{\#} = \{\mathfrak{f} \mid A \in \mathfrak{f}\}$ (A open)

$y_X : X \to 2^{X^{\mathrm{op}}}$

$m_X(\Psi) = \{\mathfrak{x} \in UX \mid Uy_X(\mathfrak{x}) \in \Psi\}$

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	cocomplete
down-set monad $\mathbb{D}$	

For Top: $X \mapsto 2^{X^{\mathrm{op}}} \simeq FX$ with basic opens $A^{\#} = \{\mathfrak{f} \mid A \in \mathfrak{f}\}$ (A open)

$y_X : X \to 2^{X^{\mathrm{op}}} \simeq FX$, $x \mapsto \mathcal{O}(x)$

$m_X(\Psi) = \{\mathfrak{x} \in UX \mid Uy_X(\mathfrak{x}) \in \Psi\}$

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	cocomplete
down-set monad $\mathbb{D}$	

For Top: $X \mapsto 2^{X^{\mathrm{op}}} \simeq FX$ with basic opens $A^{\#} = \{\mathfrak{f} \mid A \in \mathfrak{f}\}$ (A open)

$y_X : X \to 2^{X^{\mathrm{op}}} \simeq FX$, $x \mapsto \mathcal{O}(x)$

$m_X(\Psi) = \{\mathfrak{x} \in UX \mid Uy_X(\mathfrak{x}) \in \Psi\}$ resp. $\{A \mid A^{\#} \in \Psi\}$

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	cocomplete
down-set monad $\mathbb{D}$	filter monad $\mathbb{F}$

For Top: $X \mapsto 2^{X^{\mathrm{op}}} \simeq FX$ with basic opens $A^{\#} = \{\mathfrak{f} \mid A \in \mathfrak{f}\}$ (A open)

$y_X : X \to 2^{X^{\mathrm{op}}} \simeq FX,\ x \mapsto \mathcal{O}(x)$

$m_X(\Psi) = \{\mathfrak{x} \in UX \mid Uy_X(\mathfrak{x}) \in \Psi\}$ resp. $\{A \mid A^{\#} \in \Psi\}$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	continuous lattice
down-set monad $\mathbb{D}$	filter monad $\mathbb{F}$

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	continuous lattice
down-set monad $\mathbb{D}$	filter monad $\mathbb{F}$

$\mathbb{F}$ is of Kock–Zöberlein type:

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	continuous lattice
down-set monad $\mathbb{D}$	filter monad $\mathbb{F}$

$\mathbb{F}$ is of Kock–Zöberlein type: $(F y_X \leq y_{FX})$

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	continuous lattice
down-set monad $\mathbb{D}$	filter monad $\mathbb{F}$

$\mathbb{F}$ is of Kock–Zöberlein type: $(F y_X \leq y_{FX})$

X algebra $\iff$ y_X right adjoint, X T_0 $\iff$ $\alpha \cdot y_X = 1_X$.

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	continuous lattice
down-set monad $\mathbb{D}$	filter monad $\mathbb{F}$
non-empty down-closed subset	
directed down-closed subset	

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	continuous lattice
down-set monad $\mathbb{D}$	filter monad $\mathbb{F}$
non-empty down-closed subset	
directed down-closed subset	

For $A \subseteq X$ down-closed

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	continuous lattice
down-set monad $\mathbb{D}$	filter monad $\mathbb{F}$
non-empty down-closed subset	
directed down-closed subset	

For $A \subseteq X$ down-closed

$$\mathrm{Up}(X) \to 2,\ B \mapsto [\![\exists x \in X \,.\, x \in A \ \ \&\ \ x \in B]\!] = [\![A \cap B \neq \varnothing]\!]$$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	continuous lattice
down-set monad $\mathbb{D}$	filter monad $\mathbb{F}$
non-empty down-closed subset	
directed down-closed subset	

For $A \subseteq X$ down-closed resp. $\mathfrak{f} \in FX$ $(= \mathcal{A} \subseteq UX$ closed):

$$\mathsf{Up}(X) \to 2,\ B \mapsto [\![\exists x \in X \,.\, x \in A \ \ \& \ \ x \in B]\!] = [\![A \cap B \neq \varnothing]\!]$$

$$\mathsf{Cl}(X) \to 2,\ B \mapsto [\![\exists \mathfrak{x} \in UX \,.\, \mathfrak{x} \in \mathcal{A} \ \ \& \ \ \mathfrak{x} \in UB]\!] = [\![\mathcal{A} \cap UB \neq \varnothing]\!]$$

For an ordered set X	For a topological space X
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	continuous lattice
down-set monad $\mathbb{D}$	filter monad $\mathbb{F}$
non-empty down-closed subset	proper filter of opens
directed down-closed subset	

For $A \subseteq X$ down-closed resp. $\mathfrak{f} \in FX$ $(= \mathcal{A} \subseteq UX$ closed):

$$\mathrm{Up}(X) \to 2,\ B \mapsto [\![\exists x \in X \,.\, x \in A \ \ \& \ \ x \in B]\!] = [\![A \cap B \neq \varnothing]\!]$$

$$\mathrm{Cl}(X) \to 2,\ B \mapsto [\![\exists \mathfrak{x} \in UX \,.\, \mathfrak{x} \in \mathcal{A} \ \ \& \ \ \mathfrak{x} \in UB]\!] = [\![\mathcal{A} \cap UB \neq \varnothing]\!]$$

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	continuous lattice
down-set monad $\mathbb{D}$	filter monad $\mathbb{F}$
non-empty down-closed subset	proper filter of opens
directed down-closed subset	prime filter of opens

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	continuous lattice
down-set monad $\mathbb{D}$	filter monad $\mathbb{F}$
non-empty down-closed subset	proper filter of opens
directed down-closed subset	prime filter of opens

Completely distributive:

Em Ord: X (cd) $\iff y_X \vdash \mathrm{Sup}_X \vdash t$,

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	continuous lattice
down-set monad $\mathbb{D}$	filter monad $\mathbb{F}$
non-empty down-closed subset	proper filter of opens
directed down-closed subset	prime filter of opens

Completely distributive:

Em Ord: X (cd) $\iff$ $\mathcal{y}_X \vdash \mathsf{Sup}_X \vdash t$, $\quad \mathsf{CDOrd}_{\mathsf{sup}} \simeq \mathsf{kar}(\mathsf{Rel})$.

For an ordered set X	**For a topological space X**
$x \leq y$	$\mathfrak{x} \to x$
up-closed subset	closed subset
down-closed subset	filter of opens
upper bound	limit point
supremum	smallest limit point
cocomplete	cocomplete (but not really)
cocomplete	continuous lattice
down-set monad $\mathbb{D}$	filter monad $\mathbb{F}$
non-empty down-closed subset	proper filter of opens
directed down-closed subset	prime filter of opens

Completely distributive:

Em Ord: X (cd) $\iff$ $y_X \vdash \mathrm{Sup}_X \vdash t$, $\quad \mathsf{CDOrd}_{\mathsf{sup}} \simeq \mathsf{kar}(\mathsf{Rel})$.

Em Top: X (cd) $\iff$ $y_X \vdash \mathrm{Sup}_X \vdash t$, $\quad \mathsf{CDTop}_{\mathsf{sup}} \simeq$???.

We will consider $\mathbb{T} = (T, y, m)$ being

- the filter monad $\mathbb{F}$ on Top.

We will consider $\mathbb{T} = (T, y, m)$ being

- the filter monad $\mathbb{F}$ on Top.
- the proper filter monad $\mathbb{F}_1$ on Top.

We will consider $\mathbb{T} = (T, y, m)$ being

- the filter monad $\mathbb{F}$ on Top.
- the proper filter monad $\mathbb{F}_1$ on Top.
- the prime filter monad $\mathbb{F}_\omega$ on Top.

We will consider $\mathbb{T} = (T, y, m)$ being

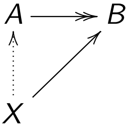
- the filter monad $\mathbb{F}$ on Top.
- the proper filter monad $\mathbb{F}_1$ on Top.
- the prime filter monad $\mathbb{F}_\omega$ on Top.
- the completely prime filter monad $\mathbb{F}_\Omega$ on Top.

We will consider $\mathbb{T} = (T, y, m)$ being

- the filter monad $\mathbb{F}$ on Top.
- the proper filter monad $\mathbb{F}_1$ on Top.
- the prime filter monad $\mathbb{F}_\omega$ on Top.
- the completely prime filter monad $\mathbb{F}_\Omega$ on Top.

Theorem (Rosebrugh and Wood, 2004). *For a monad $\mathbb{D}$ on a category* C *where idempotents split:* $\mathsf{kar}(\mathsf{C}_\mathbb{D}) \simeq \mathsf{Spl}(\mathsf{C}^\mathbb{D})$.

$(X, \alpha) \in \mathsf{Spl}(\mathsf{C}^\mathbb{D})$ whenever $\alpha \cdot t = 1_X$ for some homom. $t : X \to DX$
($\iff$ X is projective wrt. those homomorphisms which split in C)



We will consider $\mathbb{T} = (T, y, m)$ being

- the filter monad $\mathbb{F}$ on Top.
- the proper filter monad $\mathbb{F}_1$ on Top.
- the prime filter monad $\mathbb{F}_\omega$ on Top.
- the completely prime filter monad $\mathbb{F}_\Omega$ on Top.

Theorem (Rosebrugh and Wood, 2004). *For a monad $\mathbb{D}$ on a category C where idempotents split:* $\mathrm{kar}(\mathsf{C}_{\mathbb{D}}) \simeq \mathrm{Spl}(\mathsf{C}^{\mathbb{D}})$.

$(X, \alpha) \in \mathrm{Spl}(\mathsf{C}^{\mathbb{D}})$ whenever $\alpha \cdot t = 1_X$ for some homom. $t : X \to DX$
($\iff$ X is projective wrt. those homomorphisms which split in C)

Theorem (Gleason, 1954). *The projective compact Hausdorff spaces are precisely the extremely disconnected ones.*

Recall: $\mathsf{CompHaus} \simeq \mathsf{Set}^{\mathbb{U}}$,
X extremely disconnected whenever $\overline{A}$ open for every open $A \subseteq X$.

Let X be a topological space.

Let X be a topological space.

Theorem. *X is $\mathbb{F}_\omega$-algebra $\iff$ X is sober, core-compact, stable.*

SIMMONS, H. (1982), A couple of triples, *Topology Appl.* **13** (2), 201–223.

Let X be a topological space.

1. $U \ll V$ if every prime filter $\mathfrak{f}$ with $U \in \mathfrak{f}$ has a limit point in V.
2. X is core-compact if $U \in \mathcal{O}(x) \Rightarrow \exists\ V \in \mathcal{O}(x)$ with $U \ll V$.
3. X is stable if $U_i \ll V_i \Rightarrow (\bigcap_{i=1}^n V_i) \ll (\bigcap_{i=1}^n U_i)$.

Theorem. *X is $\mathbb{F}_\omega$-algebra $\iff$ X is sober, core-compact, stable.*

SIMMONS, H. (1982), A couple of triples, *Topology Appl.* **13** (2), 201–223.

Let X be a topological space.

Proposition. *X is a $\mathbb{F}_\omega$-algebra $\iff$ X is T_0, has "suprema" of prime filters, and $\mathfrak{f} \mapsto \operatorname{Sup} \mathfrak{f}$ is continuous.*

1. $U \ll V$ if every prime filter $\mathfrak{f}$ with $U \in \mathfrak{f}$ has a limit point in V.
2. X is core-compact if $U \in \mathcal{O}(x) \Rightarrow \exists\, V \in \mathcal{O}(x)$ with $U \ll V$.
3. X is stable if $U_i \ll V_i \Rightarrow (\bigcap_{i=1}^n V_i) \ll (\bigcap_{i=1}^n U_i)$.

Theorem. *X is $\mathbb{F}_\omega$-algebra $\iff$ X is sober, core-compact, stable.*

SIMMONS, H. (1982), A couple of triples, *Topology Appl.* **13** (2), 201–223.

Let X be a topological space.

Proposition. *X is a $\mathbb{F}_\omega$-algebra $\iff$ X is T_0, has "suprema" of prime filters, and $\mathfrak{f} \mapsto \operatorname{Sup} \mathfrak{f}$ is continuous.*

1. $U \ll V$ if every prime filter $\mathfrak{f}$ with $U \in \mathfrak{f}$ has a limit point in V.
2. X is core-compact if $U \in \mathcal{O}(x) \Rightarrow \exists\ V \in \mathcal{O}(x)$ with $U \ll V$.
3. X is stable if $U_i \ll V_i \Rightarrow (\bigcap_{i=1}^n V_i) \ll (\bigcap_{i=1}^n U_i)$.

Lemma. *For X core-compact:*
X is stable $\iff$ $\lim \mathfrak{f}$ is irreducible for all $\mathfrak{f}$ prime.

Theorem. *X is $\mathbb{F}_\omega$-algebra $\iff$ X is sober, core-compact, stable.*

Simmons, H. (1982), A couple of triples, *Topology Appl.* **13** (2), 201–223.

Let X be a topological space.

Proposition. *X is a $\mathbb{T}$-algebra $\iff$ X is T_0, has "suprema" of α-filters, and $\mathfrak{f} \mapsto \operatorname{Sup}\mathfrak{f}$ is continuous.*

1. $U \ll V$ if every prime filter $\mathfrak{f}$ with $U \in \mathfrak{f}$ has a limit point in V.
2. X is core-compact if $U \in \mathcal{O}(x) \Rightarrow \exists\ V \in \mathcal{O}(x)$ with $U \ll V$.
3. X is stable if $U_i \ll V_i \Rightarrow (\bigcap_{i=1}^{n} V_i) \ll (\bigcap_{i=1}^{n} U_i)$.

Lemma. *For X core-compact:*
X is stable $\iff$ $\lim \mathfrak{f}$ is irreducible for all $\mathfrak{f}$ prime.

Theorem. *X is $\mathbb{F}_\omega$-algebra $\iff$ X is sober, core-compact, stable.*

SIMMONS, H. (1982), A couple of triples, *Topology Appl.* **13** (2), 201–223.

Let X be a topological space.

Proposition. *X is a $\mathbb{T}$-algebra $\iff$ X is T_0, has "suprema" of α-filters, and $\mathfrak{f} \mapsto \operatorname{Sup} \mathfrak{f}$ is continuous.*

1. $U \ll_{\mathbb{T}} V$ if every $\mathfrak{f} \in TX$ with $U \in \mathfrak{f}$ has a limit point in V.
2. X is $\mathbb{T}$-core-compact if $U \in \mathcal{O}(x) \Rightarrow \exists\ V \in \mathcal{O}(x)$ with $V \ll_{\mathbb{T}} U$.
3. X is $\mathbb{T}$-stable if $V_i \ll_{\mathbb{T}} U_i \Rightarrow (\bigcap_{i=1}^{n} V_i) \ll_{\mathbb{T}} (\bigcap_{i=1}^{n} U_i)$.

Lemma. *For X core-compact:*
X is stable $\iff$ $\lim \mathfrak{f}$ is irreducible for all $\mathfrak{f}$ prime.

Theorem. *X is $\mathbb{F}_{\omega}$-algebra $\iff$ X is sober, core-compact, stable.*

Simmons, H. (1982), A couple of triples, *Topology Appl.* **13** (2), 201–223.

Let X be a topological space.

Proposition. *X is a $\mathbb{T}$-algebra $\iff$ X is T_0, has "suprema" of α-filters, and $\mathfrak{f} \mapsto \operatorname{Sup} \mathfrak{f}$ is continuous.*

1. $U \ll_{\mathbb{T}} V$ if every $\mathfrak{f} \in TX$ with $U \in \mathfrak{f}$ has a limit point in V.
2. X is $\mathbb{T}$-core-compact if $U \in \mathcal{O}(x) \Rightarrow \exists\ V \in \mathcal{O}(x)$ with $V \ll_{\mathbb{T}} U$.
3. X is $\mathbb{T}$-stable if $V_i \ll_{\mathbb{T}} U_i \Rightarrow (\bigcap_{i=1}^n V_i) \ll_{\mathbb{T}} (\bigcap_{i=1}^n U_i)$.

Lemma. *For X $\mathbb{T}$-core-compact:*
X is $\mathbb{T}$-stable $\iff$ $\lim \mathfrak{f}$ is irreducible for all $\mathfrak{f} \in TX$.

Theorem. *X is $\mathbb{F}_\omega$-algebra $\iff$ X is sober, core-compact, stable.*

Simmons, H. (1982), A couple of triples, *Topology Appl.* **13** (2), 201–223.

Let X be a topological space.

Proposition. *X is a $\mathbb{T}$-algebra $\iff$ X is T_0, has "suprema" of α-filters, and $\mathfrak{f} \mapsto \operatorname{Sup}\mathfrak{f}$ is continuous.*

1. $U \ll_{\mathbb{T}} V$ if every $\mathfrak{f} \in TX$ with $U \in \mathfrak{f}$ has a limit point in V.
2. X is $\mathbb{T}$-core-compact if $U \in \mathcal{O}(x) \Rightarrow \exists\ V \in \mathcal{O}(x)$ with $V \ll_{\mathbb{T}} U$.
3. X is $\mathbb{T}$-stable if $V_i \ll_{\mathbb{T}} U_i \Rightarrow (\bigcap_{i=1}^{n} V_i) \ll_{\mathbb{T}} (\bigcap_{i=1}^{n} U_i)$.

Lemma. *For X $\mathbb{T}$-core-compact:*
X is $\mathbb{T}$-stable $\iff$ $\lim \mathfrak{f}$ is irreducible for all $\mathfrak{f} \in TX$.

Theorem. *X is $\mathbb{T}$-algebra $\iff$ X is sober, $\mathbb{T}$-core-compact, $\mathbb{T}$-stable.*

SIMMONS, H. (1982), A couple of triples, *Topology Appl.* **13** (2), 201–223.

Let X be a $\mathbb{T}$-algebra with structure $\mathsf{Sup} : TX \to X$.

Let X be a $\mathbb{T}$-algebra with structure $\mathrm{Sup} : TX \to X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{x \in X \mid x = \mathrm{Sup}(\mathfrak{f}) \text{ for some } \mathfrak{f} \in TX,\ A \in \mathfrak{f}\}$.

Let X be a $\mathbb{T}$-algebra with structure $\mathrm{Sup}: TX \to X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{x \in X \mid x = \mathrm{Sup}(\mathfrak{f}) \text{ for some } \mathfrak{f} \in TX,\ A \in \mathfrak{f}\}$.
- X is called $\mathbb{T}$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{O}X$.

Let X be a $\mathbb{T}$-algebra with structure $\mathrm{Sup} : TX \to X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{x \in X \mid x = \mathrm{Sup}(\mathfrak{f}) \text{ for some } \mathfrak{f} \in TX,\ A \in \mathfrak{f}\}$.
- X is called $\mathbb{T}$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{O}X$.

For $A, B \in \mathcal{O}X$ and $(A_i)_{i \in I}$ a family of opens, $\#(I) < \alpha \in \{0, 1, \omega, \Omega\}$:

1. $A \subseteq \mu(A)$.
2. If $A \subseteq B$, then $\mu(A) \subseteq \mu(B)$.
3. $\mu(\bigcup_{i \in I} A_i) \subseteq \bigcup_{i \in I} \mu(A_i)$
4. $A \cap \mu(B) \subseteq \mu(A \cap B)$.
5. If X is $\mathbb{T}$-disconnected, then $\mu\mu(A) \subseteq \mu(A)$.
6. If X is $\mathbb{T}$-disconnected, then $\mu(A \cap B) = \mu(A) \cap \mu(B)$.

Let X be a $\mathbb{T}$-algebra with structure $\mathrm{Sup} : TX \to X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{x \in X \mid x = \mathrm{Sup}(\mathfrak{f}) \text{ for some } \mathfrak{f} \in TX,\, A \in \mathfrak{f}\}$.
- X is called $\mathbb{T}$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{O}X$.

Lemma. *If* $t \dashv \mathrm{Sup}$, *then* $t(x) = \{A \in \mathcal{O}X \mid x \in \mu(A)\}$.

Let X be a $\mathbb{T}$-algebra with structure $\mathrm{Sup} : TX \to X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{x \in X \mid x = \mathrm{Sup}(\mathfrak{f}) \text{ for some } \mathfrak{f} \in TX,\ A \in \mathfrak{f}\}$.
- X is called $\mathbb{T}$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{O}X$.

Lemma. *If* $t \dashv \mathrm{Sup}$, *then* $t(x) = \{A \in \mathcal{O}X \mid x \in \mu(A)\}$.

- If $t \dashv \mathrm{Sup}$, then $\mu(A) = t^{-1}(A^{\#})$.

Let X be a $\mathbb{T}$-algebra with structure $\mathrm{Sup} : TX \to X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{x \in X \mid x = \mathrm{Sup}(\mathfrak{f}) \text{ for some } \mathfrak{f} \in TX,\ A \in \mathfrak{f}\}$.
- X is called $\mathbb{T}$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{O}X$.

Lemma. *If $t \dashv \mathrm{Sup}$, then $t(x) = \{A \in \mathcal{O}X \mid x \in \mu(A)\}$.*

- If $t \dashv \mathrm{Sup}$, then $\mu(A) = t^{-1}(A^{\#})$.
- If X is $\mathbb{T}$-disconnected, then $t(x) := \{A \in \mathcal{O}X \mid x \in \mu(A)\}$ defines a continuous map $t : X \to TX$.

Let X be a $\mathbb{T}$-algebra with structure $\mathrm{Sup} : TX \to X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{x \in X \mid x = \mathrm{Sup}(\mathfrak{f}) \text{ for some } \mathfrak{f} \in TX,\, A \in \mathfrak{f}\}$.
- X is called $\mathbb{T}$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{O}X$.

Lemma. *If $t \dashv \mathrm{Sup}$, then $t(x) = \{A \in \mathcal{O}X \mid x \in \mu(A)\}$.*

- If $t \dashv \mathrm{Sup}$, then $\mu(A) = t^{-1}(A^{\#})$.
- If X is $\mathbb{T}$-disconnected, then $t(x) := \{A \in \mathcal{O}X \mid x \in \mu(A)\}$ defines a continuous map $t : X \to TX$.

Lemma. *If X is $\mathbb{T}$-disconnected, then t is a $\mathbb{T}$-homomorphism.*

Let X be a $\mathbb{T}$-algebra with structure $\mathrm{Sup} : TX \to X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{x \in X \mid x = \mathrm{Sup}(\mathfrak{f}) \text{ for some } \mathfrak{f} \in TX,\ A \in \mathfrak{f}\}$.
- X is called $\mathbb{T}$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{O}X$.

Lemma. *If $t \dashv \mathrm{Sup}$, then $t(x) = \{A \in \mathcal{O}X \mid x \in \mu(A)\}$.*

- If $t \dashv \mathrm{Sup}$, then $\mu(A) = t^{-1}(A^{\#})$.
- If X is $\mathbb{T}$-disconnected, then $t(x) := \{A \in \mathcal{O}X \mid x \in \mu(A)\}$ defines a continuous map $t : X \to TX$.

Lemma. *If X is $\mathbb{T}$-disconnected, then t is a $\mathbb{T}$-homomorphism.*

Lemma. *If X is $\mathbb{T}$-disconnected, then $\mathrm{Sup}(t(x)) = x$.*

Let X be a $\mathbb{T}$-algebra with structure $\mathrm{Sup} : TX \to X$.

- For $A \in \mathcal{O}X$: $\mu(A) = \{x \in X \mid x = \mathrm{Sup}(\mathfrak{f}) \text{ for some } \mathfrak{f} \in TX,\ A \in \mathfrak{f}\}$.
- X is called $\mathbb{T}$-disconnected if $\mu(A)$ is open, for every $A \in \mathcal{O}X$.

Lemma. *If $t \dashv \mathrm{Sup}$, then $t(x) = \{A \in \mathcal{O}X \mid x \in \mu(A)\}$.*

- If $t \dashv \mathrm{Sup}$, then $\mu(A) = t^{-1}(A^{\#})$.
- If X is $\mathbb{T}$-disconnected, then $t(x) := \{A \in \mathcal{O}X \mid x \in \mu(A)\}$ defines a continuous map $t : X \to TX$.

Lemma. *If X is $\mathbb{T}$-disconnected, then t is a $\mathbb{T}$-homomorphism.*

Lemma. *If X is $\mathbb{T}$-disconnected, then $\mathrm{Sup}(t(x)) = x$.*

Theorem. *X is $\mathbb{T}$-distributive $\iff$ X is $\mathbb{T}$-disconnected.*

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$

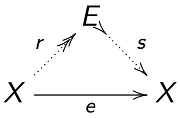
About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$

- idempotent:

$$X \xrightarrow[e]{} X \qquad\qquad e \cdot e = e$$

About $\mathrm{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$

- idempotent:

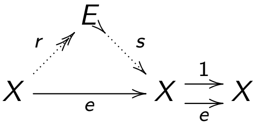


$$r \cdot s = 1_E$$

$$e \cdot e = e$$

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$

- idempotent:

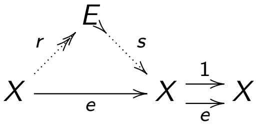


$$r \cdot s = 1_E$$

$$e \cdot e = e$$

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$

- idempotent:



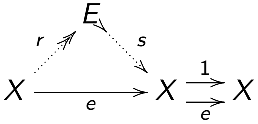
$r \cdot s = 1_E$

$e \cdot e = e$

- idempotents split in C whenever all are of this form.

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$

- idempotent:



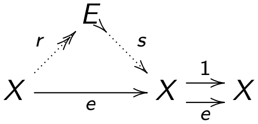
$r \cdot s = 1_E$

$e \cdot e = e$

- idempotents split in C whenever all are of this form.
- idempotents split completion can be calculated as follows:

About $\mathrm{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathrm{Spl}(\mathsf{C}^{\mathbb{T}})$

- idempotent:



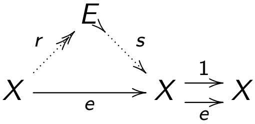
$$r \cdot s = 1_E$$

$$e \cdot e = e$$

- idempotents split in C whenever all are of this form.
- idempotents split completion can be calculated as follows:
  - embed C fully into a category X where idempotents split,

About $\mathrm{kar}(C_{\mathbb{T}}) \simeq \mathrm{Spl}(C^{\mathbb{T}})$

- idempotent:



$$r \cdot s = 1_E$$

$$e \cdot e = e$$

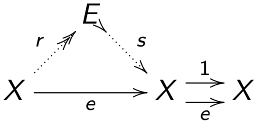
- idempotents split in C whenever all are of this form.
- idempotents split completion can be calculated as follows:
  - embed C fully into a category X where idempotents split,
  - then the idempotents split completion of C is "the closure of C in X".

$$X \in \overline{C} \;:\Longleftrightarrow\; X \text{ splits an idempotent in C}$$

$$\Longleftrightarrow\; X \text{ is a split subobject of some object in C}$$

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$

- idempotent:



$$r \cdot s = 1_E$$

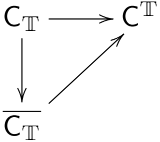
$$e \cdot e = e$$

- idempotents split in C whenever all are of this form.
- idempotents split completion can be calculated as follows:
  - embed C fully into a category X where idempotents split,
  - then the idempotents split completion of C is “the closure of C in X”.

$$X \in \overline{\mathsf{C}} \;:\Longleftrightarrow\; X \text{ splits an idempotent in C}$$

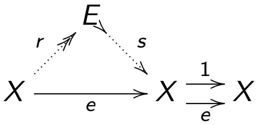
$$\Longleftrightarrow\; X \text{ is a split subobject of some object in C}$$

Example:



About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$

- idempotent:



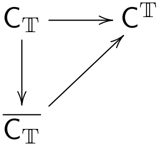
$$r \cdot s = 1_E \qquad e \cdot e = e$$

- idempotents split in C whenever all are of this form.
- idempotents split completion can be calculated as follows:
  - embed C fully into a category X where idempotents split,
  - then the idempotents split completion of C is “the closure of C in X”.

$$X \in \overline{\mathsf{C}} \;:\Longleftrightarrow\; X \text{ splits an idempotent in C}$$

$$\Longleftrightarrow\; X \text{ is a split subobject of some object in C}$$

Example:



$$\overline{\mathsf{C}_{\mathbb{T}}} \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$$

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$, $\mathbb{T} = \mathbb{F}_{\alpha}$, $\alpha \in \{0, 1, \omega, \Omega\}$, $\mathsf{C} = \mathsf{Top}$.

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$, $\mathbb{T} = \mathbb{F}_{\alpha}$, $\alpha \in \{0, 1, \omega, \Omega\}$, $\mathsf{C} = \mathsf{Top}$.

$\mathbb{T}$ is also induced by $\mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha} \overset{\eta}{\Rightarrow} \underset{\mathcal{O}}{\overset{F_{\alpha}}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathsf{Top}$,

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$, $\mathbb{T} = \mathbb{F}_{\alpha}$, $\alpha \in \{0, 1, \omega, \Omega\}$, $\mathsf{C} = \mathsf{Top}$.

$\mathbb{T}$ is also induced by $\mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha} \overset{\eta}{\Rightarrow} \underset{\mathcal{O}}{\overset{F_{\alpha}}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathsf{Top}$,

which gives fully faithful $\mathcal{O} : \mathsf{Top}_{\mathbb{T}} \to \mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha}$.

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$, $\mathbb{T} = \mathbb{F}_{\alpha}$, $\alpha \in \{0, 1, \omega, \Omega\}$, $\mathsf{C} = \mathsf{Top}$.

$\mathbb{T}$ is also induced by $\mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha} \overset{\eta}{\Rightarrow} \underset{\mathcal{O}}{\overset{F_\alpha}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathsf{Top}$,

which gives fully faithful $\mathcal{O} : \mathsf{Top}_{\mathbb{T}} \to \mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha}$.

Lemma. $L \in \overline{\mathsf{Top}_{\mathbb{T}}} \iff$ *L is a frame,* $\mathrm{hom}(L, 2)$ *separates points.*

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$, $\mathbb{T} = \mathbb{F}_\alpha$, $\alpha \in \{0, 1, \omega, \Omega\}$, $\mathsf{C} = \mathsf{Top}$.

$\mathbb{T}$ is also induced by $\mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha} \overset{\eta}{\Rightarrow} \underset{\mathcal{O}}{\overset{F_\alpha}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathsf{Top}$,

which gives fully faithful $\mathcal{O} : \mathsf{Top}_{\mathbb{T}} \to \mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha}$.

Lemma. $L \in \overline{\mathsf{Top}_{\mathbb{T}}} \iff$ *L is a frame,* $\hom(L, 2)$ *separates points.*

Theorem. $\mathsf{Frm}^{\mathrm{op}}_{\wedge,\alpha} \simeq \mathsf{Spl}(\mathsf{Top}^{\mathbb{F}_\alpha})$. *Furthermore, a topological space X is $\mathbb{F}_\alpha$-distributive if and only if X is the α-filter space of a frame.*

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$, $\mathbb{T} = \mathbb{F}_{\alpha}$, $\alpha \in \{0, 1, \omega, \Omega\}$, $\mathsf{C} = \mathsf{Top}$.

$\mathbb{T}$ is also induced by $\mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha} \overset{\eta}{\Rightarrow} \underset{\mathcal{O}}{\overset{F_\alpha}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathsf{Top}$,

which gives fully faithful $\mathcal{O} : \mathsf{Top}_{\mathbb{T}} \to \mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha}$.

Lemma. $L \in \overline{\mathsf{Top}_{\mathbb{T}}} \iff$ *L is a frame,* $\mathrm{hom}(L, 2)$ *separates points.*

Theorem. $\mathsf{Frm}^{\mathrm{op}}_{\wedge,\alpha} \simeq \mathsf{Spl}(\mathsf{Top}^{\mathbb{F}_\alpha})$. *Furthermore, a topological space* X *is* $\mathbb{F}_\alpha$*-distributive if and only if* X *is the* α*-filter space of a frame.*

$\alpha = \Omega$: Duality between spatial frames and sober spaces.

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$, $\mathbb{T} = \mathbb{F}_{\alpha}$, $\alpha \in \{0, 1, \omega, \Omega\}$, $\mathsf{C} = \mathsf{Top}$.

$\mathbb{T}$ is also induced by $\mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha} \overset{\eta}{\Rightarrow} \underset{\mathcal{O}}{\overset{F_\alpha}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathsf{Top}$,

which gives fully faithful $\mathcal{O} : \mathsf{Top}_{\mathbb{T}} \to \mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha}$.

Lemma. $L \in \overline{\mathsf{Top}_{\mathbb{T}}} \iff$ *L is a frame,* $\mathsf{hom}(L, 2)$ *separates points.*

Theorem. $\mathsf{Frm}^{\mathrm{op}}_{\wedge,\alpha} \simeq \mathsf{Spl}(\mathsf{Top}^{\mathbb{F}_\alpha})$. *Furthermore, a topological space X is* $\mathbb{F}_{\alpha}$*-distributive if and only if X is the* α*-filter space of a frame.*

$\alpha = \Omega$: Duality between spatial frames and sober spaces.

$\alpha = \omega$: Restriction of Priestley duality to frames and f-spaces.

- PRIESTLEY, H.A. (1970), Representation of distributive lattices by means of ordered Stone spaces, *Bull. London Math. Soc.* **2**, 186–190.
- PULTR, A. and SICHLER, J. (1988), Frames in Priestley's duality, *Cahiers Topologie Géom. Différentielle Catég.* **29** (3), 193–202.

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$, $\mathbb{T} = \mathbb{F}_\alpha$, $\alpha \in \{0, 1, \omega, \Omega\}$, $\mathsf{C} = \mathsf{Top}$.

$\mathbb{T}$ is also induced by $\mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha} \overset{\eta}{\Rightarrow} \underset{\mathcal{O}}{\overset{F_\alpha}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathsf{Top}$,

which gives fully faithful $\mathcal{O} : \mathsf{Top}_{\mathbb{T}} \to \mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha}$.

Lemma. $L \in \overline{\mathsf{Top}_{\mathbb{T}}} \iff$ *L is a frame,* $\mathrm{hom}(L, 2)$ *separates points.*

Theorem. $\mathsf{Frm}^{\mathrm{op}}_{\wedge,\alpha} \simeq \mathsf{Spl}(\mathsf{Top}^{\mathbb{F}_\alpha})$. *Furthermore, a topological space X is* $\mathbb{F}_\alpha$*-distributive if and only if X is the* α*-filter space of a frame.*

$\alpha = \Omega$: Duality between spatial frames and sober spaces.

$\alpha = \omega$: Restriction of Priestley duality to frames and f-spaces.

$\alpha = 0$: $\mathsf{Frm}^{\mathrm{op}}_{\wedge} \simeq \mathsf{CDTop}_{\mathsf{sup}}$

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$, $\mathbb{T} = \mathbb{F}_\alpha$, $\alpha \in \{0, 1, \omega, \Omega\}$, $\mathsf{C} = \mathsf{Top}$.

$\mathbb{T}$ is also induced by $\mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha} \overset{\eta}{\Rightarrow} \underset{\mathcal{O}}{\overset{F_\alpha}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathsf{Top}$,

which gives fully faithful $\mathcal{O} : \mathsf{Top}_{\mathbb{T}} \to \mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha}$.

Lemma. $L \in \overline{\mathsf{Top}_{\mathbb{T}}} \iff$ *L is a frame,* $\mathsf{hom}(L, 2)$ *separates points.*

Theorem. $\mathsf{Frm}^{\mathrm{op}}_{\wedge,\alpha} \simeq \mathsf{Spl}(\mathsf{Top}^{\mathbb{F}_\alpha})$. *Furthermore, a topological space X is $\mathbb{F}_\alpha$-distributive if and only if X is the α-filter space of a frame.*

$\alpha = \Omega$: Duality between spatial frames and sober spaces.

$\alpha = \omega$: Restriction of Priestley duality to frames and f-spaces.

$\alpha = 0$: $\mathsf{Frm}^{\mathrm{op}}_{\wedge} \simeq \mathsf{CDTop}_{\mathsf{sup}}$, hence

$$\mathsf{Leftadjoints}(\mathsf{Frm}_{\wedge}) \simeq \mathsf{Rightadjoints}(\mathsf{CDTop}_{\mathrm{sup}})$$

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$, $\mathbb{T} = \mathbb{F}_\alpha$, $\alpha \in \{0, 1, \omega, \Omega\}$, $\mathsf{C} = \mathsf{Top}$.

$\mathbb{T}$ is also induced by $\mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha} \overset{\eta}{\Rightarrow} \underset{\mathcal{O}}{\overset{F_\alpha}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathsf{Top}$,

which gives fully faithful $\mathcal{O} : \mathsf{Top}_{\mathbb{T}} \to \mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha}$.

Lemma. $L \in \overline{\mathsf{Top}_{\mathbb{T}}} \iff$ *L is a frame,* $\mathrm{hom}(L, 2)$ *separates points.*

Theorem. $\mathsf{Frm}^{\mathrm{op}}_{\wedge,\alpha} \simeq \mathsf{Spl}(\mathsf{Top}^{\mathbb{F}_\alpha})$. *Furthermore, a topological space X is $\mathbb{F}_\alpha$-distributive if and only if X is the α-filter space of a frame.*

$\alpha = \Omega$: Duality between spatial frames and sober spaces.

$\alpha = \omega$: Restriction of Priestley duality to frames and f-spaces.

$\alpha = 0$: $\mathsf{Frm}^{\mathrm{op}}_{\wedge} \simeq \mathsf{CDTop}_{\mathsf{sup}}$, hence

$$\mathsf{Frm} \simeq \mathsf{Leftadjoints}(\mathsf{Frm}_\wedge) \simeq \mathsf{Rightadjoints}(\mathsf{CDTop}_{\mathrm{sup}})$$

About $\mathsf{kar}(\mathsf{C}_{\mathbb{T}}) \simeq \mathsf{Spl}(\mathsf{C}^{\mathbb{T}})$, $\mathbb{T} = \mathbb{F}_{\alpha}$, $\alpha \in \{0, 1, \omega, \Omega\}$, $\mathsf{C} = \mathsf{Top}$.

$\mathbb{T}$ is also induced by $\mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha} \overset{\eta}{\Rightarrow} \underset{\mathcal{O}}{\overset{F_\alpha}{\rightleftarrows}} \overset{\varepsilon}{\Leftarrow} \mathsf{Top}$,

which gives fully faithful $\mathcal{O} : \mathsf{Top}_{\mathbb{T}} \to \mathsf{SLat}^{\mathrm{op}}_{\wedge,\alpha}$.

Lemma. $L \in \overline{\mathsf{Top}_{\mathbb{T}}} \iff$ *L is a frame,* $\hom(L, 2)$ *separates points.*

Theorem. $\mathsf{Frm}^{\mathrm{op}}_{\wedge,\alpha} \simeq \mathsf{Spl}(\mathsf{Top}^{\mathbb{F}_\alpha})$. *Furthermore, a topological space X is $\mathbb{F}_\alpha$-distributive if and only if X is the α-filter space of a frame.*

$\alpha = \Omega$: Duality between spatial frames and sober spaces.

$\alpha = \omega$: Restriction of Priestley duality to frames and f-spaces.

$\alpha = 0$: $\mathsf{Frm}^{\mathrm{op}}_{\wedge} \simeq \mathsf{CDTop}_{\mathsf{sup}}$, hence

$$\mathsf{Frm} \simeq \mathsf{Leftadjoints}(\mathsf{Frm}_{\wedge}) \simeq \mathsf{Rightadjoints}(\mathsf{CDTop}_{\mathrm{sup}}) \simeq \mathsf{CDTop}.$$