Continuous V-categories

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Motivation

D. Scott (1970)

A continuous lattice *is*

- an ordered set satisfying certain completeness properties.

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D. Scott (1970); A. Day (1975)

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- an ordered set satisfying certain completeness properties.
- an injective topological space.
- an algebra for the filter monad.

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Topological space: $\mathsf{2} = (2, \&, \mathrm{true})$, $\mathbb{U} = (U, e, m)$

X with $a : UX \times X \longrightarrow \mathsf{2}$ such that

$$\mathrm{true} \models (\dot{x} \longrightarrow x), \qquad (\mathfrak{X} \longrightarrow \mathfrak{x} \,\&\, \mathfrak{x} \longrightarrow x) \models m_X(\mathfrak{X}) \longrightarrow x.$$

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$$\begin{array}{ccc} X & \xrightarrow{e_X} & UX \\ & \overset{\leq}{{}_{1_X}\searrow} & \big\downarrow{\scriptstyle a} \\ & & X \end{array} \qquad\qquad \begin{array}{ccc} UUX & \xrightarrow{m_X} & UX \\ {\scriptstyle Ua}\big\downarrow & \leq & \big\downarrow{\scriptstyle a} \\ UX & \xrightarrow[a]{} & X. \end{array}$$

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Back to ordered sets

Suprema (part I)

$$\mathrm{Sup}_X : PX \longrightarrow X$$

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down-closed set

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$$\begin{array}{rcll} \text{down-closed set} & = & X^{\mathrm{op}} \longrightarrow 2 & (= X \,{-\!\!\circ\!\!\rightarrow}\, 1) \\ \text{up-closed set} & = & X \longrightarrow 2 & (= 1 \,{-\!\!\circ\!\!\rightarrow}\, X) \end{array}$$

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$$\begin{array}{ccc} X & \xrightarrow[\circ]{\leq} & X \\ {\scriptstyle \varphi}\downarrow\circ & & \\ 1 & & \end{array}$$

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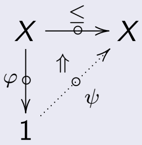

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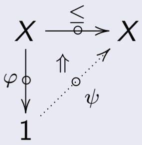

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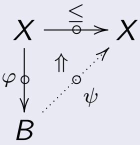

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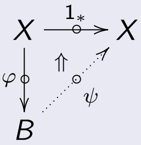

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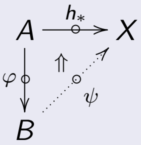

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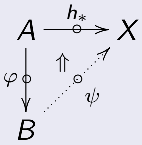

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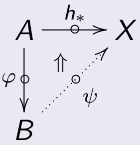

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Returning to Top (resp. $(\mathbb{T}, \mathsf{V})$-Cat)

Definition

A V-relation $\varphi : TX \relbar\joinrel\mapsto Y$ is a distributor $\varphi : X \relbar\joinrel\circ\joinrel\rightharpoonup Y$ between $X = (X, a)$ and $Y = (Y, b)$ whenever $b \circ \varphi = \varphi$ and $\varphi \circ a = \varphi$.

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- The forgetful functor $(\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cocont}_{\mathrm{sep}} \longrightarrow \mathsf{Set}$ is monadic.

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we should relativise everything with respect to a class Φ of distributors satisfying:

(Ax 1). For each $(\mathbb{T}, \mathsf{V})$-functor f, $f^* \in \Phi$.

(Ax 2). Φ is closed under certain compositions.

(Ax 3). For all $\varphi : X \multimap Y$: $(\forall y \in Y \,.\, y^* \circ \varphi \in \Phi) \Rightarrow \varphi \in \Phi$.

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- $(\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cocont}^{\Phi}_{\mathrm{sep}} \longrightarrow \mathsf{Set}$ *is monadic if* Φ *satisfies also*

  (Ax 4). *For each surjective* $(\mathbb{T}, \mathsf{V})$*-functor* f, $f_* \in \Phi$.

Following [EF99]

Examples

- Φ the class of all distributors.
- $\Phi = \{f^* : X \multimap Y\}$ the class of all representable distributors.

M. Escardó and B. Flagg.
Semantic domains, injective spaces and monads.
Electronic Notes in Theoretical Computer Science 20 (1999).

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